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OBLIQUE DEVIATION AND REFRACTION PRODUCED BY PRISMS.

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The occurrence in the MONTHLY of a few problems[1] in geometrical optics indicates that some of its readers are interested in this subject. As the field is a fairly fertile one for applied mathematics, work in it deserves to be encouraged. Since, in general, there are many discouraging difficulties and pitfalls that the mathematical reader will encounter if he attempts to gain information first hand from the monographs and advanced treatises on geometrical optics, it occurred to the present writer that a relatively short essay dealing with oblique refraction by prisms might be of some interest and perhaps value in the pages of the MONTHLY.

It is but fair to state, at the outset, that none of the results obtained—save the last one—is new. On the other hand, most of the proofs are original and they have been developed with special reference to brevity and rigor.

The only physical law that will be needed is called Snell's law. The complete statement of this law involves two facts which are embodied respectively in the sentence: "The angles of incidence and refraction are coplanar," and in the formula:

$$n_1 \sin a_1 = n_2 \sin a_2. \tag{1}$$

If the ray of light passes from medium 1 into medium 2, then a_1 and a_2 denote respectively the angles of incidence and refraction. n_1 and n_2 symbolize the absolute indices of refraction for the first and second media. The "absolute" index means the ratio of the velocity of light in empty space to the velocity of light in the material dispersive medium in question. The ratio $n_2/n_1 \equiv n$ is called the "relative" index of refraction of medium 2 with respect to medium 1.

In Fig. 1 let the plane IZ suggest the first face of the prism, and let FI and IS indicate respectively the incident ray and the refracted ray. Air and glass may be imagined at the left and right of this plane respectively. Stated broadly, our problem will be to investigate briefly the behavior of the emergent ray beyond the second face of the prism (intentionally omitted from Fig. 1) as the incident ray FI is moved around the point of incidence, I, into all possible positions.

Experience has shown that it is not advantageous to determine the angular positions of the rays by employing a rectangular coördinate frame, direction

[1] THE AMER. MATH. MONTHLY, vol. 26, 80, 1919; vol. 27, 35, 1920.

cosines, component vectors, etc. Instead, the analysis and the resulting theorems are very appreciably simplified by introducing a "principal plane," NJ, and by projecting the rays orthogonally on this plane.

Definitions: The line in which the two refracting planes or faces intersect is called the "refracting edge" of the prism. The angle between the refracting planes is named the "refracting angle." Any plane passing through the prism perpendicular to the refracting edge is termed a "principal plane." Hence, a principal plane is parallel to the bases of a right prism of solid geometry. The orthogonal projection, GIG_1, on a principal plane of the oblique ray, FIS, is called the "projected ray." Although both the oblique ray and the projected ray are mathematical fictions,[1] it is conducive to clearness of thought to treat the former as a physical reality (that is, the path along which the energy is transmitted in a non-crystalline medium), and the latter as a purely geometric auxiliary line. The angles which the oblique ray and the projected ray beyond the second or emergence face of the prism make with the corresponding rays before incidence at the first face are called respectively the "deviation of the (oblique) ray" and the "deviation of the projected ray," or the "oblique deviation" and the "projected deviation." The angles, v_1, v_2, $\cdots$,[2] which the segments of the complete oblique ray make with the orthogonal projections on a principal plane will be called the "altitudes" of the rays. Similarly the angles, h_1, h_2, $\cdots$,[2] which the segments of the complete projected ray form with the normals, $NIN_1, \cdots$, to the faces of the prism, at the two points where refraction occurs, will be called the "azimuths" of the oblique ray.

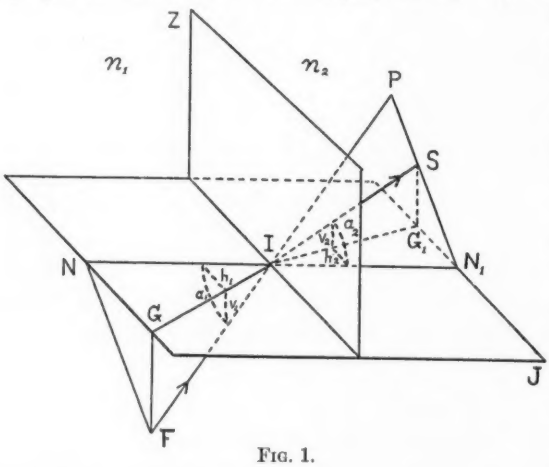


FIG. 1.

In Fig. 1, $NIF \equiv a_1$ and $N_1IS \equiv a_2$ denote respectively the angle of incidence and the angle of refraction of the oblique ray (at the first face of the prism). The outline $FNIN_1P$ indicates the plane of incidence at the first face. The mathematical reader should now be prepared to follow the rest of the paper without undue distraction arising from justifiable unfamiliarity with the highly specialized subject.

For the sake of both consistency and brevity we shall not prove the first formulas of oblique refraction by adding certain highly artificial construction

[1] The wave-length of light is quite finite, diffraction phenomena exist, etc.

[2] h and v to suggest horizontal and vertical respectively.

lines to (the definitional) Fig. 1 and by employing certain theorems of solid geometry (as is done in the older[1] treatises); instead we shall make use of a far more powerful, elegant and general method. This method consists in moving the rays (which are unlocalized vectors), without changing their relative angular directions, until they radiate from the center of a sphere of unit radius. The points where the rays intersect the spherical surface may then be used to define the directions of the rays, and the problems will be reduced primarily to the realm of spherical trigonometry.

Let us now imagine a spherical diagram drawn with the equatorial circle representing the principal plane passing through the point of incidence. Let the points N_1, P, and S indicate respectively the normal to the first face of the prism at the point of incidence, the incident ray, and the refracted ray. Let H mark the pole of the equatorial circle on the same side of the plane of the circle as the points P and S. Imagine great circle arcs connecting H to the points P and S. Let the points of intersection of these two arcs with the equatorial circle be denoted by p and s respectively.

Then the arcs N_1p and N_1s measure the azimuths h_1 and h_2 of the incident and refracted rays. Similarly the arcs pP and sS measure the corresponding altitudes v_1 and v_2.

If a great circle arc be imagined connecting the points N_1 and P it will also pass through the point S because the angle of refraction must lie in the same plane as the angle of incidence (Snell's law in part). Hence the arcs N_1P and N_1S measure respectively the angle of incidence, a_1, and the angle of refraction, a_2.

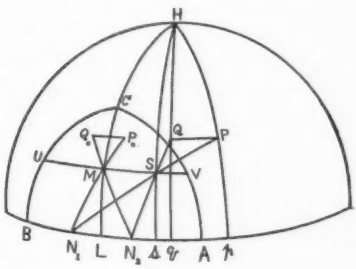


FIG. 2.

We shall now prove that

$$n_1 \sin v_1 = n_2 \sin v_2, \qquad (2)$$

$$n_1 \cos v_1 \sin h_1 = n_2 \cos v_2 \sin h_2. \qquad (3)$$

In the right spherical triangles N_1pP and N_1sS each member of the following equation is equal to the sine of the common angle pN_1P

$$\frac{\sin v_1}{\sin a_1} = \frac{\sin v_2}{\sin a_2}.$$

Combining this result with equation (1) we obtain formula (2), at a glance. Thus we have the theorem: *The altitudes of the incident and refracted rays obey the trigonometric part of the law of refraction.* Obviously, the angles v_1 and v_2 are not coplanar.

In the right triangles used above, the tangent of the angle pN_1P gives the following equation

$$\frac{\tan v_1}{\sin h_1} = \frac{\tan v_2}{\sin h_2}$$

[1] *Cf.* R. S. Heath, *Geometrical Optics*, p. 21, 1887.

or

$$\sin v_2 \cos v_1 \sin h_1 = \sin v_1 \cos v_2 \sin h_2.$$

Combining this equation with formula (2) we obtain immediately formula (3). If $n_1 \cos v_1$ and $n_2 \cos v_2$ be replaced by n_1' and n_2' respectively, equation (3) assumes the form

$$n_1' \sin h_1 = n_2' \sin h_2,$$

which is algebraically of the same type as formulæ (1) and (2). By analogy, it is customary to speak of $n_1 \cos v_1$ and $n_2 \cos v_2$ as "effective" absolute indices of refraction. This is appropriate since there is no justifiable danger of confusing effective indices with true ones, and n_1' and n_2' play an important rôle in many problems of advanced geometrical optics. This being understood, we may now state the theorem: *The azimuths of the incident and refracted rays obey the law of refraction with effective indices whose ratio is proportional to the ratio of the cosines of the corresponding altitudes.* In this case the planes of the angles h_1 and h_2 coincide.[1]

We are now prepared to follow the path of the oblique ray through and beyond the complete prism.

The points N_2 and Q represent respectively the normal to the second face of the prism at the point of emergence, and the emergent ray. By Snell's law N_2, S, and Q lie on the same great circle. $N_2S \equiv a_3$ is the angle of incidence (within the prism) at the second face. $N_2Q \equiv a_4$ is the angle of refraction of the emergent ray. $qQ \equiv v_4$ is the altitude of the emergent ray, while $N_2q \equiv h_4$ is the azimuth of the same. $N_2s \equiv h_3$ is the azimuth of the incident ray at the second face. $N_1N_2 \equiv \beta$ is the refracting angle of the prism. $QP \equiv D$ is the total deviation of the oblique ray. $qp \equiv D'$ is the total deviation of the projected ray. Angles in the plane N_1sp will be defined as positive when their initial sides have to be rotated in a counterclockwise direction—as viewed from H—in order to bring them into coincidence with the terminal sides. None of the angles measured from normals can exceed $\pi/2$.

Assuming the same medium (usually air) to be in contact with both faces of the prism, formula (2), when applied to the second or emergence face, becomes[2]

$$n_1 \sin v_4 = n_2 \sin v_2. \tag{4}$$

Inspection of equations (2) and (4) shows that $v_4 = v_1$, therefore: *When a prism is surrounded by a single medium, the incident and emergent rays make equal angles with their orthogonal projections on a principal plane.*[3]

It should be noted that this result is independent of the values of the indices

[1] Since only one refracting plane was involved in the preceding proofs it should be clear that the theorems are not restricted to prisms or even to plane interfaces. In Fig. 1, say, IZ may be the tangent plane to any smooth curved surface at the point of incidence. Also the projected ray is a possible path for light of wave-length different from that along the oblique ray.

[2] Obviously $v_3 \equiv v_2$.

[3] This would not hold for the element of a compound prism having air in contact with one face and a different kind of glass with the other face.

of refraction n_1 and n_2. Also it is just what we might have expected, since the elements of the surfaces of the prism which are perpendicular to the principal plane act as a plane-parallel slab of refracting material for the vertical (parallel to the refracting edge) components of the rays, and a slab of this shape (in a single medium) produces no deviation but only a parallel displacement of the rays.

Consequently, in Fig. 2, $qQ = pP = v_1$, and the triangle PHQ is isosceles. Since $\angle\, qHp = D'$, $QP = D$, and $PH = \pi/2 - v_1$ we may imagine a great circle drawn through H so as to bisect QP (and qp). Either of the resulting halves of the triangle PHQ gives

$$\sin \tfrac{1}{2}D = \sin \tfrac{1}{2}D' \cos v_1. \tag{5}$$

Now

$$qp = N_1p - (N_2q + N_1N_2)$$

or

$$D' = h_1 - h_4 - \beta. \tag{6}$$

But, as stated above, $|h_1| \not> \pi/2$, $|h_4| \not> \pi/2$, hence $|h_1 - h_4| \not> \pi$, and relation (6) shows that $D'/2$ must be acute ($\beta > 0$).

For $v_1 \neq 0$, $\cos v_1 < 1$; hence formula (5) shows that $D < D'$ or: *The total deviation of an oblique ray by a triangular prism surrounded by a single medium is less than that of the orthogonal projection of the ray on a principal plane.*

This accounts for the fact that the monochromatic images of a rectilinear slit (spectral lines) formed by any prism spectroscope or spectrograph are appreciably curved. This phenomenon may be observed by looking through any[1] prism toward the vertical boundary of a window, the refracting edge of the prism being vertical also, of course.

Before proceeding farther with the properties of the deviations, attention should be called to a condition which must be fulfilled in order that light may be transmitted directly by a triangular prism. The greatest value that the angle of incidence a_1 can attain is $\pi/2$ ("grazing" incidence) and, when this is the case, equation (1) shows that the greatest value of the angle of refraction a_2 is given by

$$a_2 = \sin^{-1}\frac{1}{n} \equiv c,$$

where $n \equiv n_2/n_1 > 1$. This limiting value of a_2 is called the "critical" angle, c. Similarly, in order that the ray may emerge from the second face of the prism, a_3 must not exceed c. Accordingly, if arcs of small circles be described with N_1 and N_2 as centers and with spherical radii of length c, a lune-like area will be enclosed having the property that the point S,—representative of the internal ray,—must lie within, or on the boundary of, this region in order that transmission without total reflection may occur.[2] In Fig. 2, the upper half of this area is bounded by the arcs AC and BC. If the lower intersection of the limiting small circles be designated as C', then the arcs $C'AC$ and $C'BC$ correspond respectively to grazing incidence and grazing emergence. The points C and C' each signify simultaneous grazing incidence and emergence.

[1] The refracting angle must not be so large as to preclude transmission.

[2] For $n = 1.5$, $c = 41°\ 48'\ 37''$, and the greatest value of $\beta = 2c = 83°\ 37'\ 14''$.

The most complete and satisfactory method of exploring the domain of transmission $CBC'AC$, for minima of D and D', etc., is to proceed in two steps: (a) to keep the altitude of the ray incident upon the first face of the prism constant while varying the azimuth, and (b) to follow out the behavior of the deviations corresponding to the salient features found under (a) while suitably varying the altitude of the incident ray. More precisely: (a) to keep v_2 [and hence, by (2), v_1 also] constant and explore along a small circle of altitude v_2, such as UV in Fig. 2, and (b) to pick out the interesting points on UV and investigate how the deviations at these points vary as v_2 is changed in such a manner as to sweep over the entire domain from C to BA to C'. The associated analysis will be appreciably simplified by drawing an arc through H and C, cutting the principal plane in L ($N_1L = LN_2 = \beta/2$), and by introducing a new azimuthal angle x, for S, which is reckoned from L as origin. Then equation (3) gives

$$\sin h_1 = n' \sin (x + \tfrac{1}{2}\beta), \tag{7}$$

$$\sin h_4 = n' \sin (x - \tfrac{1}{2}\beta), \tag{8}$$

where

$$n' \equiv n \cos v_2/\cos v_1 = [n^2 + (n^2 - 1) \tan^2 v_1]^{1/2} > 1 \text{ (for } n > 1).$$

Now let us see what happens along UV. Keeping n' constant while differentiating equations (6), (7), and (8) with respect to x, we find

$$\frac{\partial D'}{\partial x} = \frac{\partial h_1}{\partial x} - \frac{\partial h_4}{\partial x},$$

$$\cos h_1 \frac{\partial h_1}{\partial x} = n' \cos (x + \tfrac{1}{2}\beta),$$

$$\cos h_4 \frac{\partial h_4}{\partial x} = n' \cos (x - \tfrac{1}{2}\beta),$$

whence

$$\frac{\partial D'}{\partial x} = n'[\cos (x + \tfrac{1}{2}\beta)/\cos h_1 - \cos (x - \tfrac{1}{2}\beta)/\cos h_4].$$

Omitting the limiting cases $h_1 = \pi/2$ and $h_4 = \pi/2$ we may multiply and divide by the sum of the two fractions within the bracket, thus obtaining

$$\frac{\partial D'}{\partial x} = \frac{n'[\cos^2 (x + \tfrac{1}{2}\beta)/\cos^2 h_1 - \cos^2 (x - \tfrac{1}{2}\beta)/\cos^2 h_4]}{\cos (x + \tfrac{1}{2}\beta)/\cos h_1 + \cos (x - \tfrac{1}{2}\beta)/\cos h_4}.$$

By elementary trigonometric transformations, the last equation may be readily reduced to

$$\frac{\partial D'}{\partial x} = \frac{2n'(n'^2 - 1) \sin \beta \sin x \cos x}{\cos h_1 \cos h_4(\cos h_1 \cos h_3 + \cos h_2 \cos h_4)}. \tag{9}$$

Since $\beta \not> \pi$, $n' > 1$, and none of the remaining angles can exceed $\pi/2$, we see that the sign of the derivative is the same as that of $\sin x$. For any two

permissible values of x having the same magnitude but opposite signs, the derivative also has equal numerical values with opposite signs. Moreover it vanishes when $x = 0$. Consequently, the values of the deviation of the projected ray, D', are symmetrical with respect to the great circle $x = 0$, and they have a minimum both in the algebraic and in the arithmetic sense when, and only when, $x = 0$. Inspection of equations (6), (7), and (8) shows that no exception to the last statement arises when either h_1 or h_4 equals $\pi/2$. When $x = 0$, $h_2 = -h_3 = \beta/2$, and

$$h_1 = -h_4 = \sin^{-1}(n' \sin \tfrac{1}{2}\beta).$$

Since v_1 is being kept constant, equation (5) shows that D has precisely the same minimum properties as D'.

These results may be interpreted in terms of the actual prism in the following manner. Imagine a plane diaphragm constructed in the interior of the prism and fulfilling three conditions; (a) of passing through the point of incidence at the first face, (b) of being parallel to the refracting edge, and (c) of making equal angles $(\pi - \beta)/2$ with the refracting faces of the prism, that is, crossing the prism symmetrically. Then, as the altitude of the ray incident upon the first face is kept constant while its azimuth is varied, the oblique deviation and the projected deviation will both have minimum values at the instant when the internal ray lies in the plane of symmetry just defined. The winglike planes containing the minimum positions of the incident and emergent rays, which may be imagined outside of the prism, will also be symmetrically situated, but the common value of the equal angles, $\pi/2 - \sin^{-1}[n' \sin(\beta/2)]$, that these planes make with the refracting faces of the prism will not be independent of the altitude of the incident ray, hence their positions will be altered if a new constant value for v_1 be taken, since n' is a function of v_1.

With this interpretation of the term "symmetrical," we may now state the theorem: *When the altitude of the internal ray, or of the external rays, is kept constant, both the oblique deviation and the projected deviation have least values when, and only when, the rays are situated symmetrically with respect to the refracting faces of the prism.*

For the sake of completeness it may be remarked that, when the altitude of the incident ray is kept constant while the azimuth is varied, the incident ray and the internal ray each describe portions of the lateral surfaces of right circular cones having as common axis a line (in the incidence face) passing through the point of incidence and parallel to the refracting edge, the semi-apical angles of the cones being respectively the complements of the altitudes v_1 and v_2.

Let D_0 and D_0' denote respectively the minimum values of D and D' which occur when x equals zero and v_1 is constant. The next step will be to keep x zero and to investigate the behavior of the (partial) minima D_0 and D_0' as the point S moves along the great circle arc from C to L to C'. The independent variable is now v_2. With direct reference to the prism, our immediate problem is to cause the altitudes of the three parts of the complete ray to vary in such a

manner as to maintain the above mentioned symmetry and to investigate the way in which the minima of deviation depend upon the values of the altitudes. Under these circumstances the internal ray will be constrained to move up or down (fanwise) in the transverse diaphragm introduced earlier. It will be convenient to take up D_0 before D_0'.

Consider the halves of the isosceles triangles N_1MN_2 and P_0MQ_0, and let each of the four equal acute angles having the common vertex M be denoted by e. Since $Q_0P_0/2 = D_0/2$, $N_1L = \beta/2$, $MP_0 = a_1 - a_2$, and $N_1M = a_2$, we have

$$\sin e = \sin \tfrac{1}{2}D_0/\sin (a_1 - a_2) = \sin \tfrac{1}{2}\beta/\sin a_2$$

or

$$\sin \tfrac{1}{2}D_0 = (\sin a_1 \cos a_2 - \cos a_1 \sin a_2) \sin \tfrac{1}{2}\beta/\sin a_2.$$

As equation (1) is

$$\sin a_1 = n \sin a_2$$

it is clear that

$$\cos a_1 = + [n^2 \cos^2 a_2 - (n^2 - 1)]^{1/2}$$

and

$$\sin \tfrac{1}{2}D_0 = \{n \cos a_2 - [n^2 \cos^2 a_2 - (n^2 - 1)]^{1/2}\} \sin \tfrac{1}{2}\beta. \tag{10}$$

As $LM = v_2$, the right triangle N_1LM gives

$$\cos a_2 = \cos \tfrac{1}{2}\beta \cos v_2, \tag{11}$$

which, combined with the process of rationalizing the braced expression in equation (10), leads to

$$\sin \tfrac{1}{2}D_0 = \frac{(n^2 - 1) \sin \tfrac{1}{2}\beta}{n \cos \tfrac{1}{2}\beta \cos v_2 + [(n \cos \tfrac{1}{2}\beta \cos v_2)^2 - (n^2 - 1)]^{1/2}}. \tag{12}$$

Since $\cos v_2$ increases as v_2 decreases numerically, we see from equation (12) that D_0 attains its least value when $v_2 = 0$, that is, when the entire ray lies in the principal plane. This does not show explicitly that D_0 has a minimum in the algebraic sense when $v_2 = 0$, for, a cusp of the first species might be present at the lowest point of a surface constructed on a rectangular coördinate frame and exhibiting D as a function of x and v_2.

In order to show that D_0 has a true algebraic minimum when $v_2 = 0$ we must form the derivative of D_0 with respect to v_2. It will be found, without difficulty, by differentiating equations (10) and (11), and then eliminating da_2/dv_2, that

$$\frac{dD_0}{dv_2} = \frac{n \sin \beta \sin v_2}{\cos \tfrac{1}{2}D_0} \left\{ \frac{n \cos \tfrac{1}{2}\beta \cos v_2}{[(n \cos \tfrac{1}{2}\beta \cos v_2)^2 - (n^2 - 1)]^{1/2}} - 1 \right\}.$$

The expression enclosed by braces in the last equation, $\sin \beta$, and $\cos \tfrac{1}{2}D_0$ are always positive, hence the derivative has the same sign as $\sin v_2$. The derivative also approaches zero simultaneously with v_2. It should be evident, therefore, that D_0 has an algebraic minimum when, and only when, $v_2 = 0$.

When $x = 0$, equation (5) may be written

$$\sin \tfrac{1}{2}D_0' = \sin \tfrac{1}{2}D_0 \sec v_1,$$

which shows, at a glance, that D_0' has an algebraic minimum simultaneously with D_0, for both $\sin(D_0/2)$ and $\sec v_1$ decrease as v_2 becomes smaller.

The whole region bounded by $CBC'AC$ has now been explored and we have proved the theorem: *The deviation produced by direct transmission through a triangular prism has the smallest value, which is also an absolute algebraic minimum, when the ray lies in a principal plane and makes equal angles with the refracting faces.* This is subject to the qualifications that the prism is surrounded by one medium and that its relative index of refraction is greater than unity.

As far as the writer is aware, the following generalization of formula (5) is entirely new. Suppose we have k prisms with their refracting edges parallel, and surrounded by one and the same medium. The prisms must be so situated, of course, as to permit the passage of light through the system.

The hypothesis of a single surrounding medium causes v_1 to be constant throughout the entire prism train, quite independently of the various indices of refraction of the materials composing the different prisms. [See equation (4) and the associated context.] Also, the refracting angles need not bear any particular relation to one another. Since the emergent ray from any prism (save the last) becomes the incident ray of the next succeeding prism, the representative points P and Q of Fig. 2 will be distributed along a small circle arc of altitude v_1. Accordingly, the separate deviations produced by the individual prisms will form collectively a fluting of great circle arcs having salient points on the small circle just mentioned. With these flutings, however, we are not directly concerned, for the single great circle arc connecting the terminus of the last or kth emergent ray to that of the first incident ray will represent the total deviation D produced by the entire train of prisms. Incidentally, $D < \sum_1^k D_j$.

As in the case of a single prism, so also here, the triangle PHQ,—where P and Q now mark the extreme ends of the fluting,—will be isosceles, so that

$$\sin \tfrac{1}{2}D = \sin \tfrac{1}{2}D' \cos v_1. \qquad (5')$$

Since the segments of the oblique ray are all projected orthogonally on the common principal plane, the following sub-relation between the total projected deviation and the separate projected deviations obviously holds:

$$D' = \sum_1^k D_j'.$$

BIBLIOGRAPHICAL AND HISTORICAL NOTES: The most reliable and elegant presentation of geometrical optics in English is the second edition of J. P. C. Southall's *Principles and Methods of Geometrical Optics*, New York (1913). The most complete general discussion of prisms was written by H. Konen and published in H. Kayser's *Handbuch der Spectroscopie*, vol. I, pp. 253–394, (1900). On page 258 may be found a very complete list of references to proofs of the theorem of minimum deviation *in a principal plane*.

This theorem dates from the work of Newton. The reference given by Konen

is "Lect. opt. London 1729. P. I., Sect. II, Art. 31." For oblique deviation, the fact that $v_4 = v_1$ was published by A. Bravais in 1845, *Journal de l'Ecole Polytechnique*, vol. 18, 30^e Cahier, p. 79. The priority for formula (5) seems to belong to Mascart. This equation is given correctly in his *Traité d'Optique*, vol. 1, p. 84 (1889). By drawing an incorrect diagram, R. S. Heath (*l.c.* p. 32) derived the formula

$$\cos \tfrac{1}{2}D = \cos \tfrac{1}{2}D' \cos v_1.$$

This was copied by almost all later writers regardless of the fact that J. Larmor called attention to the error in the *Proceedings of the Cambridge Philosophical Society*, vol. 9, p. 108 (1896). An unbiased discussion of this matter (including the part which the present writer has taken) is given in Southall's treatise, p. 127. Finally, Konen in attempting to generalize formula (5) extended the error even farther by stating that the cosine equation still held (*l.c.* p. 267).

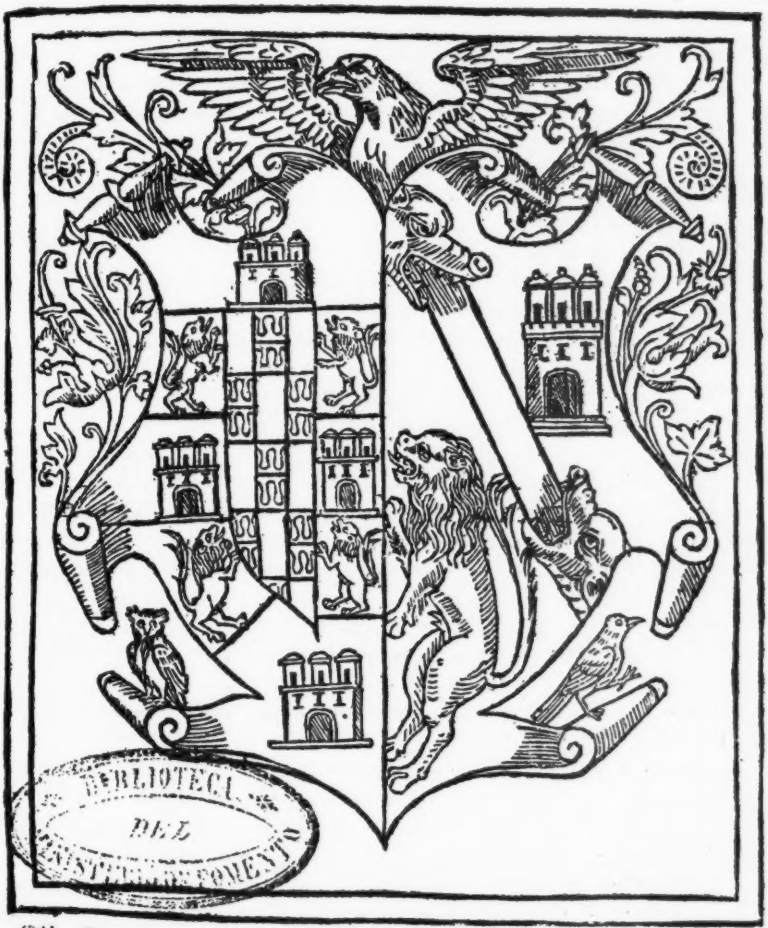
THE FIRST WORK ON MATHEMATICS PRINTED IN THE NEW WORLD.[1]

By DAVID EUGENE SMITH, Columbia University.

I. General Description. If the student of the history of education were asked to name the earliest work on mathematics published by an American press, he might, after a little investigation, mention the anonymous arithmetic that was printed in Boston in the year 1729. It is now known that this was the work of Isaac Greenwood who held for some years the chair of mathematics in what was then Harvard College. If he should search the records still further back, he might come upon the American reprint of Hodder's well-known English arithmetic, the first textbook on the subject, so far as known, to appear in our language on this side of the Atlantic. If he should look to the early Puritans in New England for books of a mathematical nature, or to the Dutch settlers in New Amsterdam, he would look in vain; for, so far as known, all the colonists in what is now the United States were content to depend upon European textbooks to supply the needs of the relatively few schools that they maintained in the seventeenth century.

The earliest mathematical work to appear in the New World, however, antedated Hodder and Greenwood by more than a century and a half. It was published long before the Puritans had any idea of migrating to another continent, and fifty years before Henry Hudson discovered the river that bears his name. Of this work, known as the *Sumario Compendioso*, there remain perhaps only four copies, and it is desirable, not alone because of its rarity but because of its im-

[1] Address delivered before The Mathematical Association of America, and the section on the History of Science of the American Association for the Advancement of Science, at the University of Chicago, December 28, 1920. The extracts are from a fac-simile reprint of the original work soon to published by Ginn & Company, Boston, with translation and notes by Professor Smith.


Sumario cõpẽdioso delas quẽtas
de plata y oro q̃ en los reynos del Piru son necessarias a
los mercaderes: y todo genero de tratantes. Cõ algunas
reglas tocantes al Aritbmetica.
Fecho por Juan Diez freyle.

Title page of the first work on mathematics printed in the New World, Mexico City, 1556.

portance in the history of education on the American continent, that some record of its contents should be made known to scholars.

In order to understand the *Sumario Compendioso* it is necessary to consider briefly the political and social situation in Mexico in the middle of the sixteenth century. Cortés entered the ancient city of Tenochtitlan, later known as Mexico, in the year 1519, but its capture and destruction occurred two years later, in 1521. Thus, in the very year that Luther was attacking certain ancient customs and privileges in the Old World, the representatives of other ancient customs and privileges were attacking and destroying a worthy civilization in the newly discovered continent.

The first viceroy of New Spain, which included the present Mexico, was a man of remarkable genius and of prophetic vision,—Don Antonio de Mendoza. He assumed his office in 1535, and for fifteen years administered the affairs of the colony with such success as to win for himself the name of "the good viceroy." He founded schools, established a mint, ameliorated the condition of the natives, and encouraged the development of the arts. In his efforts at improving the condition of the people he was ably assisted by Juan de Zumárraga, the first Bishop of Mexico. Among the various activities of these leaders was the arrangement made with the printing establishment of Juan Cromberger of Seville whereby a branch should be set up in the capital of New Spain.

As a result of this arrangement there was sent over as Cromberger's representative one Juan Pablos, a Lombard printer, and so the "casa de Juan Cromberger" was established, prepared to spread the doctrines of the Church to the salvation of the souls of the unbelievers. Cromberger himself never went to Mexico, but his name appears either on the *portadas* or in the colophons of all the early books. From and after 1545, however, the name is no longer seen, Cromberger having already died in 1540.

The author of the *Sumario* was one Juan Diez, a native of the Spanish province of Galicia, a companion of Cortés in the conquest of New Spain, and the editor of the works of Juan de Avila, "the apostle of Andalusia," and of the *Itinerario* of the Spanish fleet to Yucatan in 1518. He is sometimes confused with another Juan Diaz (the name being spelled both ways), a contemporary theologian and author. In a letter written to Charles V in 1533 he is mentioned as a "clérigo anciano y honrado," so that he must have been advanced in years when the *Sumario* appeared. That this was the case is also apparent from a record of the expedition of 1518 in which it is stated that "triximus vn clerigo que dezia joan diaz," doubtless a young and adventurous apostle, full of zeal and desire to make known the gospel in the New World.

Juan Diez undertook the work primarily for the purpose of assisting those who were engaged in the buying of the gold and silver which was already being taken from the mines of Peru and Mexico for the further enriching of the moneyed class and the rulers of Spain. He felt that he could best serve this purpose by preparing such a set of tables as should relieve these merchants as far as possible from any necessity for computation. Apparently, however, he was prompted by

the further demand for a brief treatment of arithmetic which should be suited to the needs of apprentices in the counting houses of the New World, and so he devotes eighteen pages to the subject of computation and presents it in a manner not unworthy of the European writers of the period.

The most interesting feature of the work, however, is neither the tables nor the arithmetic; it consists of six pages devoted to algebra, chiefly relating to the quadratic equation.

The book consists of one hundred and three folios, generally numbered. After the dedication (folios i, *v*, and ij, *r*) there is an elaborate set of tables, including those relating to the purchase price of various grades of silver (folio iij, *v*), to per cents (folio xlix, *r*), to the purchase price of gold (folio lvij, *v*), to assays (folio lxxxj, *r*), and to monetary affairs of various kinds. The mathematical text (folio xcj, *v*) consists of twenty-four pages besides the colophon (folio ciij, *v*). As already stated, eighteen of these pages relate chiefly to arithmetic, and six to algebra.

The book was printed in the City of Mexico in the year 1556, being the first work on mathematics to be printed outside the boundaries of Europe, except for the ancient block books of China.

In order to give some idea of the general nature of the work, a few of the problems will be set forth, chiefly those which illustrate the application of algebra as we understand the term today.

II. Typical Problems not listed under Algebra. 1. I bought 10 varas of velvet at 20 pesos less than cost, for 34 pesos plus a vara of velvet. How much did it cost a vara?

Rule: Add 20 pesos to 34 pesos, making 54 pesos, which will be your dividend. Subtract one from 10 varas, leaving 9. Divide 54 by 9, giving 6, the price per vara.

Proof: 10 varas at 6 pesos is 60 pesos. This minus 20 pesos is 40. You paid 34 pesos plus a vara costing 6 pesos, and this gives the result, 40 pesos.

2. I bought 9 varas of velvet for as much more than 40 pesos as 13 varas at the same price is less than 70 pesos. How much did a vara cost?

Rule: Add the pesos, 40 and 70, making 110. Add the varas, 9 and 13, making 22. Dividing 110 by 22 the quotient is 5, the price of each vara.

Proof: 9 varas at 5 pesos is 45 pesos, which is 5 more than 40 pesos; and 13 varas at 5 pesos is 65, which is 5 pesos less than 70, as you see.

3. Required a number which if 8 is added to it will be a square, and if 8 is subtracted from it will also be a square. Take half of eight, which is 4; square it, making 16; add 1, making 17, and this is the number to which if you add 8 you have 25, the root of which is 5; and if 8 is taken from it there is left 9, the root of which is 3; for 3 times 3 is 9, as you see.

4. Find 2 numbers the sum of the squares of which will make a square number which has an integral root. The first numbers are 3 and 4, for their squares are 9 and 16, and these added together make 25, the root of which is 5. Observe that

you have 5 numbers; the first are 2 and 3; the next are 3 and 4, the proposed numbers; and there is also 5, which is their root. Place these numbers as you see in the figure below. Then use cross multiplication, saying "3 times 3 is 9, and 2 times 4 is 8." Place these numbers at the right-hand side, one under the other. Then multiply again at the top, 2 times 3 is 6; and underneath, 3 times 4 is 12. Now subtract the less from the greater, that is, 6 from 12, and there remains 6. Divide this by 5, the root of the assumed numbers, and the quotient is $1\frac{1}{5}$, one of the numbers required. Now add 8 and 9, the products of the first multiplication, and the sum is 17. Divide this by 5 and the quotient is $3\frac{2}{5}$, and this is the second required number.

Proof: The square of $1\frac{1}{5}$ is $1\frac{1}{2}\frac{1}{5}$; the square of $3\frac{2}{5}$ is $11\frac{1}{2}\frac{4}{5}$; and these added together, as you see, make 13.

$$\begin{array}{cccccccc} & & 6 & & & & 02 & 1 \\ 5 & 2 & 6 & 3 & 9 & 17 & 17 \mid 3\tfrac{2}{5} & 6 \mid 1\tfrac{1}{5} \\ & 3 & & 4 & 8 & & 5 & 5 \\ & & 12 & & & & & \end{array}$$

III. Typical Problems listed under Algebra. Although the above problems are solved by arithmetical rules, they are essentially algebraic. Under the title *Arte Mayor* the author gives a number of examples generally involving quadratic equations, of which the following are types:

1. Find a square from which if $15\frac{3}{4}$ is subtracted the result is its own root.

Rule: Let the number be *cosa* (x). The square of half a *cosa* is equal to $\frac{1}{4}$ of a *zenso* (x^2). Adding 15 and $\frac{3}{4}$ to $\frac{1}{4}$ makes 16, of which the root is 4, and this plus $\frac{1}{2}$ is the root of the required number.

Proof: Square the square root of 16, plus half a *cosa*, which is four and a half, giving 20 and $\frac{1}{4}$, which is the square number required. From $20\frac{1}{4}$ subtract 15 and $\frac{3}{4}$ and you have 4 and $\frac{1}{2}$, which is the root of the number itself.

2. A man takes passage in a ship and asks the master what he has to pay. The master says that it will not be any more than for the others. The passenger on again asking how much it would be, the master replies: "It will be the number of pesos which, multiplied by itself and added to the number, will give 1260." Required to know how much the master asked.

Rule: Let the cost be a *cosa* of pesos. Then half of a *cosa* squared makes $\frac{1}{4}$ of a *zenso*, and this added to 1260 makes 1260 and a quarter, the root of which less $\frac{1}{2}$ of a *cosa* is the number required. Reduce 1260 and $\frac{1}{4}$ to fourths; this is equal to $\frac{5041}{4}$, the root of which is 71 halves; subtract from it half of a *cosa* and there remains 70 halves, which is equal to 35 pesos, and this is what was asked for the passage.

Proof: Multiply 35 by itself and you have 1225; adding to it 35, you have 1260, the required number.

3. A man is selling goats. The number is unknown except that it is stated that a merchant asked how many there were and the seller replied: "There are

so many that, the number being squared and the product quadrupled, the result will be 90,000." Required to know how many goats he had.[1]

A DETERMINATION OF THE CURVE MINIMIZING THE AREA ENCLOSED BY IT AND ITS EVOLUTE.

By OTTO DUNKEL, Washington University, St. Louis, Mo.

One of the problems treated in the calculus of variations as an example in which the second derivative appears in the integrand is that of determining the curve in a plane passing through two fixed points which with its evolute and its two normals of given directions at these two points enclose a minimum area.[2] The special form of the integrand permits this problem to be solved without resorting to the general theory of the calculus of variations, and the conditions appear as necessary and sufficient simultaneously. The solution given here appears to be new so far as the writer can learn from the references consulted. It is adapted only to this special form of the integrand, but it is quite possible that there may be other problems of this form to which it applies. For example, the problem of determining the curve which with its caustic produced by parallel rays of light, and the reflected rays at two of its given points, enclose a minimum area may also be solved in the same way.[3] It will be observed that the method admits of a slight generalization.

The Necessary and Sufficient Conditions for a Minimum. Let one of the fixed points be the origin and the other (x_2, y_2) and let the inclination of the curve to the y-axis at these two points be θ_1 and θ_2, respectively, $\theta_2 > \theta_1$. Let s be the length of the arc of the curve measured from the origin and, R, the radius of curvature. Then

$$R = \frac{ds}{d\theta} \tag{1}$$

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[1] In the above brief extracts the archaic forms of expression have been retained so far as the circumstances of translation permit. No effort has been made to explain, in this presentation, the method of attacking the problems, or to consider the sources from which the author drew his materials.

[2] This problem is discussed in many well-known works, for example: I. Todhunter, *Researches in the Calculus of Variations*, London, 1871, chapter 13; H. Hancock, *Lectures on the Calculus of Variations*, Cincinnati, 1904, pp. 75–76; A. Kneser, *Lehrbuch der Variationsrechnung*, Braunschweig, 1900, pp. 203, 219; and O. Bolza, *Vorlesungen über Variationsrechnung*, Leipzig, 1909, p. 152. It originated with Euler (*Methodus Inveniendi Lineas Curvas* . . . Lausannæ & Genevæ, 1744, pp. 64–66): "Invenire curvam Am, quæ cum sua evoluta AR & radio osculi mR in quavis loco applicato, minimum spatium ARm includat"; the solution which Euler gives employs the calculus of variations. In *Annals of Mathematics*, new series, vol. 14, pp. 14–26, 1912, E. J. Miles discusses the "Determination of the constants in Euler's problem concerning the minimum area between a curve and its evolute," and in this MONTHLY, *1917*, 420–422, P. R. Rider discusses "An intrinsic equation solution of a problem of Euler."

[3] A paper on this problem was read by Professor Dunkel before the American Mathematical Society, November 27, 1920.—EDITOR..

determines R as a positive quantity, as we shall suppose that the curve is always concave downward. We shall also assume that R is a continuous function of θ. The integral for the area, S, between the curve, its evolute, and the two normals at the end points

$$S = \frac{1}{2}\int_{\theta_1}^{\theta_2} R^2 d\theta \tag{2}$$

is to be made a minimum subject to the conditions

$$x_2 = \int_{\theta_1}^{\theta_2} R \sin\theta \, d\theta, \qquad y_2 = \int_{\theta_1}^{\theta_2} R \cos\theta \, d\theta. \tag{3}$$

The problem is handled more easily by replacing these by equivalent conditions involving similar integrals equated to zero. This is easily accomplished by setting

$$R = A \sin\theta + B \cos\theta + \eta, \tag{4}$$

where A and B are constants to be determined so that

$$\begin{aligned} x_2 &= A\int_{\theta_1}^{\theta_2} \sin^2\theta \, d\theta + B\int_{\theta_1}^{\theta_2} \sin\theta\cos\theta \, d\theta, \\ y_2 &= A\int_{\theta_1}^{\theta_2} \sin\theta\cos\theta \, d\theta + B\int_{\theta_1}^{\theta_2} \cos^2\theta \, d\theta, \\ 0 &= \int_{\theta_1}^{\theta_2} \eta \sin\theta \, d\theta, \qquad 0 = \int_{\theta_1}^{\theta_2} \eta \cos\theta \, d\theta. \end{aligned} \tag{4'}$$

The last two of these four equations follow from the first two and (3) and (4). The first pair of equations can always be solved for A and B, since their determinant is never zero. This may be seen as follows: the integral

$$\int_{\theta_1}^{\theta_2} (\lambda \sin\theta + \cos\theta)^2 \, d\theta$$

is a quadratic form in λ which is always positive; also the coefficient of λ^2 is always positive. Hence its discriminant is positive; but this discriminant is the determinant of the pair of equations in (4').[1] The equation (2) is now replaced by

$$S = \frac{1}{2}\int_{\theta_1}^{\theta_2} (A \sin\theta + B \cos\theta)^2 \, d\theta + \frac{1}{2}\int_{\theta_1}^{\theta_2} \eta^2 \, d\theta, \tag{2'}$$

where two integrals are omitted since they are each zero by virtue of the new conditions on η in the last line of (4'). Since R is continuous, η is continuous, and it is immediately obvious that $\eta = 0$ is a necessary and a sufficient condition

[1] It may be shown by computation that the value of the determinant is $[(\theta_2 - \theta_1)^2 - \sin^2(\theta_2 - \theta_1)]/4 > 0$, but the above method is given in preference on account of its generality. It should be noticed that the method does not depend upon the special functions $\sin\theta$ and $\cos\theta$.

for a minimum. This value of η satisfies the two equations in (4'). Hence the curve for which

$$R = A \sin\theta + B \cos\theta, \tag{5}$$

where A and B are given by (4'), gives a minimum area.

The Minimizing Curve and its Evolute. The equations for x and y differ from those in the first two lines of (4') by having θ, x, y in place of θ_2, x_2, y_2. Since not both x_2 and y_2 are zero, A and B cannot both be zero and hence we may set $A = 4a\cos\alpha$, $B = 4a\sin\alpha$, $\psi = \alpha + \theta$, where $4a = \sqrt{A^2 + B^2}$. Inserting these in (5) and rotating the axes through the angle α, we find $R = 4a\sin\psi$, where ψ is the angle that the tangent to the curve makes with the new y'-axis. The integration of the new differential equations,

$$dx' = 4a\sin^2\psi d\psi, \qquad dy' = 4a\sin\psi\cos\psi d\psi,$$

gives

$$\begin{aligned} x' + a[2(\theta_1 + \alpha) - \sin 2(\theta_1 + \alpha)] &= a(2\psi - \sin 2\psi), \\ y' + a[1 - \cos 2(\theta_1 + \alpha)] &= a(1 - \cos 2\psi). \end{aligned} \tag{6}$$

Hence the curve is a cycloid generated by a circle of radius a rolling on a straight line parallel to the x'-axis and at a distance $a[1 - \cos 2(\theta_1 + \alpha)]$ below it with a cusp at $x' = -a[2(\theta_1 + \alpha) - \sin 2(\theta_1 + \alpha)]$ and at intervals of $2\pi a$. The angle through which the circle has rolled is given by 2ψ.

If σ is the length of arc of the evolute, the radius of curvature of the evolute is

$$\rho = \frac{d\sigma}{d\psi} = 4a\cos\psi = 4a\sin\left(\frac{\pi}{2} + \psi\right).$$

The values of ψ which make R vanish determine the cusps of the cycloid and hence these cusps must be points of the evolute. Moreover at such points of the evolute ρ has its maximum absolute value $4a$. This shows that the evolute is an equal cycloid having the same direction and with its vertices at the cusps of the former.[1]

Fixed End Points and a Variable Slope at One End. We now consider the case in which the end points are fixed, the slope at the initial point is given but that at the final point is not assigned. If there is a curve C which gives a minimum area, a necessary condition may be obtained as follows. Suppose that the inclination of C at the terminal point is θ_2 and let us consider only those curves $[C]$ which have this terminal inclination and satisfy the remaining conditions. Since the area given by C does not exceed that resulting from any one of the curves $[C]$, the previous work shows that the curve C must be a cycloid. It remains now to determine the terminal inclination θ_2 of a cycloid which satisfies all the other end conditions and which gives a smaller area than any cycloid

[1] This result was given by Huygens in *Christiani Hvgenii . . . Horologivm Oscillatorivm . . .*, Paris, 1673, page 67, prop. VI: "Semicycloidis evolutione, à vertice cœpta, alia semicyclois describitur evolutæ æqualis & similis, cujus basis est in ea recta quæ cycloidem evolutam in vertice contingit."—EDITOR.

having a different terminal inclination. This is merely a problem of minimizing a definite function S of θ_2. Returning to the equations (2), (4′), (5) the expression for the area becomes

$$2S = A^2 \int_{\theta_1}^{\theta_2} \sin^2\theta\, d\theta + 2AB \int_{\theta_1}^{\theta_2} \sin\theta\cos\theta\, d\theta + B^2 \int_{\theta_1}^{\theta_2} \cos^2\theta\, d\theta, \tag{7}$$

$$= Ax_2 + By_2,$$

where A and B are functions of θ_2. Hence

$$2\frac{dS}{d\theta_2} = x_2\frac{dA}{d\theta_2} + y_2\frac{dB}{d\theta_2}. \tag{8}$$

From (4′) we have

$$0 = \frac{dA}{d\theta_2}\int_{\theta_1}^{\theta_2} \sin^2\theta\, d\theta + \frac{dB}{d\theta_2}\int_{\theta_1}^{\theta_2} \sin\theta\cos\theta\, d\theta + A\sin^2\theta_2 + B\sin\theta_2\cos\theta_2,$$

$$0 = \frac{dA}{d\theta_2}\int_{\theta_1}^{\theta_2} \sin\theta\cos\theta\, d\theta + \frac{dB}{d\theta_2}\int_{\theta_1}^{\theta_2} \cos^2\theta\, d\theta + A\sin\theta_2\cos\theta_2 + B\cos^2\theta_2. \tag{9}$$

By multiplying the first of these equations by A, the second by B, and adding the results, we have, after using (4′) and (5)

$$x_2\frac{dA}{d\theta_2} + y_2\frac{dB}{d\theta_2} + R_2^2 = 0,$$

where R_2 is the radius of curvature of the cycloid at (x_2, y_2). Hence

$$2\frac{dS}{d\theta_2} = -R_2^2, \tag{10}$$

and therefore the area decreases as θ_2 increases. There is therefore no finite value of θ_2 rendering S a minimum. We may say, however, that for a portion of a single arch the cycloid having a cusp at the terminal point gives a smaller area than any cycloid having a smaller terminal inclination. From this it follows that the portion of a single arch of a cycloid having a terminal cusp gives a smaller area than any other curve having the same or a smaller inclination at the terminal point.

If we suppose now that the inclination at the terminal point is given, but that at the initial point is variable, it will be found that

$$2\frac{dS}{d\theta_1} = R_1^2, \tag{10′}$$

where R_1 is the radius of curvature of the cycloid at the initial point. Hence the area decreases as θ_1 decreases.

Consider now the case in which the inclination at each end point is variable. The area S decreases as θ_1 decreases and θ_2 increases. Suppose for simplicity

that $y_2 = 0$; then a cycloid with the respective inclinations 0 and π at the end points, *i.e.*, a single arch with cusps at the end points, gives a smaller area than any portion of an arch with intermediate inclinations at these same end points. It thus follows that the area given by a single complete arch of a cycloid is smaller than that given by any other curve whose initial inclination is not smaller than 0 and whose terminal inclination is not greater than π.

The method employed above applies to integrals of the form (2) with any number of conditions such as (3) in which other functions of θ as well as the trigonometric may appear in the integrand. Certain light restrictions are to be placed upon such functions.

QUESTIONS AND DISCUSSIONS.

EDITED BY W. A. HURWITZ, Cornell University, Ithaca, N. Y.

REPLIES.

34 [*1917*, 134, 341; *1920*, 114, 301, 405, 460]. Given the mixed integral and functional equation

$$\int_{x=0}^{x=h} f(x)dx = \frac{h}{6}\left[f(0) + 4f\left(\frac{h}{2}\right) + f(h) \right],$$

to determine the function $f(x)$. This equation is of rather fundamental practical value as it has to do with the most general solid whose volume is given by the prismatoid formula.

I. REMARKS BY J. P. BALLANTINE, Pennsylvania State College.

In remarks by the editor on this question [*1920*, 302], attention was called to the related equation

$$\int_{\alpha}^{\beta} f(x)dx = \frac{\beta - \alpha}{6}\left[f(\alpha) + 4f\left(\frac{\alpha + \beta}{2}\right) + f(\beta) \right], \tag{1}$$

where both α and β are allowed to vary; and it was pointed out that if $f(x)$ is a solution of this equation possessing a fifth derivative, then $f(x)$ must be a polynomial of degree $\leqq 3$. It will be proved here that the same conclusion follows from the assumption that $f(x)$ is merely continuous.

I. *Let x_0, x_2, x_4, x_6 be four equally spaced points, and x_1, x_3, x_5 the points of bisection of the three equal intervals formed. Then any integrable function which satisfies* (1) *and vanishes at x_0, x_2, x_4, x_6 vanishes also at x_1, x_3, x_5.*

Let $x_2 - x_0 = 2a$. We find from (1)

$$\int_{x_0}^{x_4} f(x)dx = 0, \qquad \int_{x_2}^{x_6} f(x)dx = 0.$$

Hence, with the notation

$$\int_{x_2}^{x_4} f(x)dx = G, \tag{2}$$

we find

$$\int_{x_0}^{x_2} f(x)dx = -G, \tag{3}$$

$$\int_{x_4}^{x_6} f(x)dx = -G; \tag{4}$$

and therefore, by addition,

$$\int_{x_0}^{x_6} f(x)dx = -G + G - G = -G. \tag{5}$$

By (1) and (2),

$$G = \tfrac{1}{3}af(x_3);$$

by (1) and (5),

$$-G = 4af(x_3);$$

hence

$$G = 0, \qquad f(x_3) = 0.$$

Finally, by (1) and (3),

$$\tfrac{1}{3}af(x_1) = -G = 0,$$

and by (1) and (4),

$$\tfrac{1}{3}af(x_5) = -G = 0;$$

so that

$$f(x_1) = f(x_3) = f(x_5) = 0,$$

as stated.

II. *If an integrable function $f(x)$ satisfying* (1) *vanishes at four equally spaced points, it vanishes also at all points obtainable by repeated bisection of the intervals between the points.*

This is proved by repeated application of I.

III. *If a continuous function $f(x)$ satisfying* (1) *vanishes at four equally spaced points, it vanishes at every point of the interval bounded by the extreme points.*

Since the set of vanishing points obtained in II is everywhere dense, the identical vanishing of $f(x)$ follows at once from the added hypothesis of continuity.

IV. *Any continuous function $f(x)$ satisfying* (1) *is a polynomial of degree $\leqq 3$.*

Choose at random four equally spaced points x_0, x_2, x_4, x_6. It is possible to find $\varphi(x) = A + Bx + Cx^2 + Dx^3$, so that $\varphi(x_0) = f(x_0)$, $\varphi(x_2) = f(x_2)$, $\varphi(x_4) = f(x_4)$, $\varphi(x_6) = f(x_6)$; for on writing out these conditions, we have for the determination of A, B, C, D four linear algebraic equations whose determinant does not vanish. Then the function $f_1(x) = f(x) - \varphi(x)$ satisfies (1) and vanishes at x_0, x_2, x_4, x_6. Therefore, by III, $f_1(x)$ vanishes, $x_0 \leqq x \leqq x_6$. That is, $f(x)$ is a polynomial of degree $\leqq 3$ in the interval from x_0 to x_6. By choosing x_0, x_2, x_4, x_6 properly, any real value of x may be included. It is also easily seen that widening the interval cannot alter the values of A, B, C, D; for, if such alteration were possible, we should have two different polynomials identically equal in the smaller interval, which is impossible.

II. Remarks by Louis Weisner, New York City.

The solution of this problem, published in the Monthly, *1920*, 301–2, is based on the assumption that $f(x)$ is analytic; this assumption leading to the conclusion that "$f(x)$ can be at worst a cubic polynomial." The following solution, based upon a different assumption, leads to a more general result. Assume that

$$f(x) = c_0 + c_1x^{r_1} + c_2x^{r_2} + \cdots + c_nx^{r_n} + \cdots,$$

in which the r's are to be found, if they exist, the only restriction placed upon them being that they may not be negative. Then $f(0) = c_0$.

Substituting for $f(x)$ in the proposed equation, transposing and collecting terms, we find that c_0 disappears and that

$$\sum_{n=1}^{n=\infty}\left[c_n\left(\frac{1}{6}\cdot\frac{4}{2^{r_n}}+\frac{1}{6}-\frac{1}{r_n+1}\right)h^{r_n+1}\right] = 0.$$

This equation will be satisfied if we choose the r's so that they are the roots of the auxiliary equation

$$\frac{1}{6}\cdot\frac{4}{2^r}+\frac{1}{6}-\frac{1}{r+1} = 0,$$

or of the equation

$$(r+1)(2^{-r+2}+1) = 6.$$

It is seen by inspection that there are no negative or fractional roots of the auxiliary equation; the only positive integral roots are $r = 1$, $r = 2$, $r = 3$. However, the auxiliary equation has in fact an infinite number of roots, real or complex.

Consequently, we have

$$f(x) = c_0 + c_1x + c_2x^2 + c_3x^3 + \sum_{n=4}^{n=\infty} c_nx^{r_n},$$

in which the r's are the incommensurable or complex roots of the auxiliary equation. The c's are independent arbitrary constants, since no assumption was made regarding them.

III. Remarks by the Editor.

Mr. Ballantine's result, dealing not with the given equation but with the similar one in which both limits of integration are allowed to vary, is of considerable interest, as it not only shows that the hypothesis of continuity alone suffices to restrict solutions to cubic polynomials, but also effects the proof in a very simple and elementary way. It will be remembered that in virtue of the previous remarks of Professor Bennett and the editor [*1920*, 462], a similar result is not obtainable for the equation as stated in the question, even with the hypothesis of a continuous first derivative, and that it remains undecided what number of derivatives between one and six will suffice.

Mr. Weisner, seeking solutions other than cubic polynomials, is led to a result practically equivalent to that stated by the editor [*1920*, 463]. This result was passing through the press at the time of receipt of Mr. Weisner's manuscript. It may be observed that the editor's statement shows that in Mr. Weisner's notation, when $n > 3$, r_n is necessarily of the form $2 + \beta i$, where β is a solution of a certain transcendental equation.

DISCUSSIONS.

The two discussions in this number deal with questions of analytic geomet y. Professor Borger shows how curves may frequently be plotted advantageously by geometric construction of the points rather than by computation. He illustrates the method by a number of examples, both in rectangular and in polar coördinates, and also in parametric form. It is fairly clear that this method may at times be more useful and at other times much less useful than the usual plan. Surely the student should begin learning as early as possible that the mere laborious plotting of points, either arithmetically or geometrically, is generally to be used only as an incidental aid in determining the form of the curve, the important information being obtained from a functional study of the equation, first by the machinery of algebra, later also by that of the calculus.

Professor Bradshaw gives an interesting exposition of the inaccuracy of the figures usually given in the text-books on solid analytic geometry. The reader who has never before given attention to this matter will scarcely credit, until after direct examination, the uniformity of this error in our usual texts.

I. On Some Geometric Methods for Curve Tracing.[1]

By R. L. Borger, Ohio University.

In beginning courses in analytic geometry, curve tracing is confined almost exclusively to the process of making a table and plotting the points of the curve from the computed coördinates. This destroys in a great measure the geometric aspect of the problem, and at the same time develops in the student a quality of dependence upon the table even in cases where a geometric treatment would be simpler. For many curves in polar form, plotting from a table becomes excessively onerous. If the radius vector is a trigonometric function of the vectorial angle, the interposition of two tables becomes necessary. The student finds too that he has no means of detecting the character of a curve as he has in rectangular coördinates. Such simple curves as

[1] Read before the Mathematical Association of America, Ohio Section, Columbus, April 2, 1920.

$$\rho = a \cdot \sec\theta, \qquad \rho = 2a \cdot \cos\theta, \qquad \rho = a \cdot \sec^2\frac{\theta}{2},$$

do not reveal themselves to him unless perhaps he transforms their equations to the rectangular form, and, in this degree, he regards the polar treatment as inconvenient if not redundant. There are some cases in which this defect may be remedied by a geometric process. It is the purpose of this paper to show that if $\rho = P(f_1, f_2, f_3, f_4, f_5, f_6) \equiv P(f), f_i$ being any one of the six trigonometric functions, and $P(f)$ any polynomial in these functions, then a geometric representation of ρ may be effected for any θ.

By means of the Peano construction for curves in rectangular coördinates, and by means of well-known artifices most of the curves ordinarily occurring may be plotted without the use of a table. If the rectangular equation is of the form $f(x, y) = 0$, a parametric representation enables us to employ the method indicated above for polar coördinates. This unifies the treatment in the various representations of a curve and also gives the student greater power in the study of curve tracing. The paper contains:

1. Some methods for the study of curves in polar coördinates, and their application to curves in parameter form.
2. The Peano construction.
3. The construction of some curves.

To exhibit a construction of $\rho = P(f)$, P being a polynomial in the six trigonometric functions, we show a construction for $\rho = a \cdot f_i^n(\theta)$, n being a positive integer and f_i any one of the trigonometric functions.

1. The product $\rho = a \cdot \sin^n\theta$ (n being any integer).

Let OX be the initial line, $OA = a$ a perpendicular to it, and OP a ray such that the angle $XOP = \theta$. Draw $AP_1 \perp OP$, $P_1P_2 \perp OA$, $P_2P_3 \perp OP$, $P_3P_4 \perp OA$, and so on. Then

$$OP_1 = a \cdot \sin\theta, \qquad OP_2 = a \cdot \sin^2\theta,$$

and in general

$$OP_n = a \cdot \sin^n\theta.$$

With OX, OA, OP as before, draw $AP_1 \perp OA$, $P_1P_2 \perp OP$, $P_2P_3 \perp OA$, and so on. Then

$$OP_1 = a \cdot \csc\theta, \qquad OP_2 = a \cdot \csc^2\theta, \qquad \text{and} \qquad OP_n = a \cdot \csc^n\theta.$$

Combining these results we have

$$OP_n = a \cdot \sin^n\theta \ (n \text{ being any integer}).$$

The construction of the product $\rho = a \cdot \cos^n\theta$ is immediate. We measure $OA = a$ on the x-axis and project alternately on OP and OA; while for $\rho = a \sec^n\theta$ we erect perpendiculars alternately to OA and OP.

To construct $\rho = a \tan^n\theta$, lay off $OA = a$ on OX. Draw $AQ \perp OA$ cutting OP in Q and $QP_1 \perp OY$ cutting OY in P_1; draw $P_1P_2 \perp OP$ cutting OX in P_2.

Repeating this set of operations, we have $OP_n = a \tan^n \theta$; a slight change shows that we may obtain $a \cot^n \theta$ also. Thus the equation $\rho = a \cdot \tan^n \theta$, may be constructed when n is any integer. The construction of any cross product is obvious, and it follows that if $\rho = P(f)$ is a polynomial in the six trigonometric functions ρ may be constructed geometrically.

2. We turn now to the consideration of rectangular coördinates and the Peano[1] construction.

(*a*) To construct the product $y = F_1(x) \cdot F_2(x)$, $F_1(x)$, $F_2(x)$ being represented for any value of x by the points F_1, F_2, select the x-unit OU, and the y-unit OV, and draw the diagonal $y = x$. Let M be the foot of the ordinate through F_1. Translate either point (say F_1) parallel to the x-axis to the unit line in F_1'; and the other (F_2) to the diagonal in F_2'. Draw through F_2' a perpendicular to the x-axis cutting it in M' and cutting OF_1' in Q. Draw $QP \perp F_1M$ cutting it in P. From the similar triangles $OM'Q$ and OUF_1' it may be seen that $M'Q = MP$ is the product F_1F_2. Changing the order of translation a check on the product may be found.

(*b*) To construct the reciprocal $y = 1/F_2$.

As before, let M be the foot of the ordinate through F_2, and let U, V be unit points on the axes. Translate F_2 parallel to the x-axis to the diagonal line at F_2', draw the ordinate through F_2', meeting the horizontal unit-line at S and the x-axis at M', and draw OS meeting the vertical unit-line at P'; translate P' horizontally to P on F_2M.

From the similar triangles OUP' and $OM'S$

$$UP'/OU = M'S/OM' \qquad \text{or} \qquad y = 1/F_2 = MP.$$

By means of (*a*) and (*b*) the quotient $y = F_1/F_2$ may be at once found.

3. The Construction of Curves. (*a*) The class $y = a \cdot x^n$. Multiplication of the abscissa of each point on $y = x$ by x, by means of (*a*) above, gives the curve $y = x^2$; and the higher powers may be found by a repetition of the process. By (*b*) we construct the reciprocal of x, and the curve $y = 1/x = x^{-1}$. From this we get the nth power, and thus the curve $y = x^n$, n being any integer. A mere change in the y-scale is sufficient to give the curves $y = a \cdot x^n$, desired.

(*b*) The curves $y = a \cdot x^2 + b \cdot x + c$. The method in (*a*) enables us to get the curve $y = a \cdot x$ and a shift of the x-axis in a direction indicated by the sign of b, gives the graph of $y = a \cdot x + b$. Multiplying this by x and repeating the shift gives $y = a \cdot x^2 + b \cdot x + c$. These may be condensed into a single step.

The construction of $y = P(x)$, and of $y = P_1(x)/P_2(x)$ may now be effected.[2]

(*c*) For the curve $y = a^x$, we may construct a scale for all integral values of x and plot such points.

(*d*) The inverse functions may be graphically obtained by revolving the graph of the given function about the line $y = x$. By this means then we secure the

[1] Cf. Lunn, "Outline of a coherent course in college algebra," AMERICAN MATHEMATICAL MONTHLY, *1905*, 123–129.

[2] Cf. Runge, *Graphical Methods*, pp. 6–7.

graphs of the inverse trigonometric functions; from (*c*) the logarithmic curve, and from (*a*) the curves $y = x^{1/n}$ and further $y = x^{m/n}$.

By a transformation to polar coördinates, or by a parameter representation for curves of the form $F(x, y) = 0$, we may frequently find a convenient construction. We shall now consider some such cases.

1. $\rho = a \cdot \sec \theta$, $\rho = a \cdot \sec (\theta - \alpha)$. Any point P on the locus 1_1 is located by erecting a perpendicular to the initial line at A. Its intersection with the radius vector corresponding to the vectorial angle θ gives a point on the locus. Since these perpendiculars all coincide the points for all θ's must be on the straight line AP. In 1_2 the initial line makes an angle α with OA, and we have the general equation of the straight line.

2. $\rho = 2a \cdot \cos \theta$. Take $OA = 2a$; the projection of OA upon the various radii vectors makes it evident that the curve is a circle of diameter $2a$.

3. $\rho = a \sec^2 \theta/2$. Take $OA = a$ on the initial line; let $AOP = \theta$, $AOQ = \frac{1}{2}\theta$. Draw $AQ \perp OA$, meeting OQ in Q, and $QP \perp OQ$, meeting OP in P. Then the point P is a point of the locus, which may be seen, as follows, to be a parabola.[1]

At D on OA, making $OA = AD$, erect the perpendicular DR. Prolong OQ to meet DR in R. The triangle $OPQ = RPQ$, PR is perpendicular to DR, and $OP = PR$. Since P is any point of the locus it is a parabola. The line QP may be shown to be a tangent line, and the construction of a tangent to the curve is obvious; while the following theorems may be readily proved.

(*a*) The angle between two tangents is half the angle between the focal radii drawn to the point of contact.

(*b*) The perpendicular upon a tangent from the focus meets the tangent at the point of intersection of the tangent at the vertex.

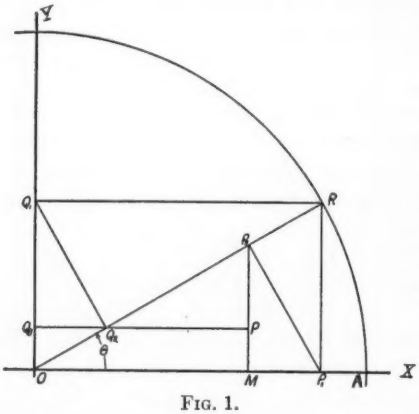

FIG. 1.

(*c*) Tangents at the extremities of a focal chord meet at right angles.

(*d*) Tangents to a parabola from a point on the directrix are perpendicular.

(*e*) Two perpendicular tangents meet on the directrix.

4. $\rho = 2a \cdot \cos 2\theta \cdot \sec \theta$. Let $OA = 2a$, the angle $AOP = \theta$, $AOR = 2\theta$. The perpendicular AR upon OR meets the terminal line of θ in P a point of the curve. If we increase θ by $\pi/2$, OR remains invariant and the prolongation of AP intersects the terminal line of the new θ, $\theta + (\pi/2)$, in another point P' of the curve. A is easily seen to be a point of the locus. Denoting by Q the intersection of PP' and OY, the angle at P' is seen to be equal to the angle QOP', and since POP' is a right angle Q is equidistant from P and P'. Thus the curve is the strophoid.

[1] Cf. Smith and Gale, *New Analytic Geometry*, p. 123.

Curves such as those given by the equations below are readily plotted by this method.

(*a*) $\rho = a \cdot \sin^n \theta$.

(*b*) $\rho = a \cdot \sin n\theta$.

(*c*) $\rho = a \cdot \tan^n \theta$, etc.

The method of geometric multiplication employed in the case of polar coördinates frequently becomes serviceable for curves in implicit form given in rectangular coördinates after representing the curve in parameter form.

$x^{2/3} + y^{2/3} = a^{2/3}$; $x = a \cdot \cos^3 \theta$, $y = a \cdot \sin^3 \theta$. Corresponding to the parameter θ, the projection of OR (Fig. 1) upon OX gives $OP_1 = a \cdot \cos \theta$, $OP_2 = a \cdot \cos^2 \theta$ and $OM = a \cdot \cos^3 \theta$. Thus $x = OM$; and similarly for $y = MP$. The equations show that at the same time points may be located in all four quadrants.

The curves $x^2y = 4a^2(2a - y)$ with the parameter form

$$x = 2a \cdot \tan \theta,$$

$$y = 2a \cdot \cos^2 \theta,$$

and $y^2 = x^3/(2a - x)$ with the parameter form

$$x = 2a \cdot \cos^2 \theta,$$

$$y = 2a \cdot \cos^2 \theta \cdot \cot \theta,$$

furnish good examples.

As a final illustration we consider the curve given in parameter form:[1]

$$x = 2r \cdot \cos \theta + r \cdot \cos 2\theta - r,$$

$$y = 2r \cdot \sin \theta - r \cdot \sin 2\theta.$$

These equations may be written in the form:

$$x = 4r \cdot \cos \theta \cdot \cos^2 \theta/2 - 2r,$$

$$y = 4r \cdot \sin \theta \cdot \sin^2 \theta/2.$$

Then the following construction may be effected (Fig. 2):

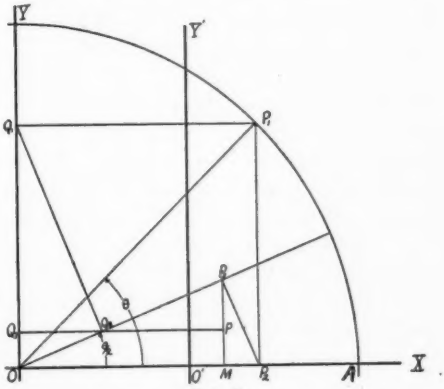

FIG. 2.

Projecting OP_1 upon the x-axis, $OP_2 = 4r \cdot \cos \theta$ ($OA = 4r$), $OP_3 = 4r \cos \theta \cdot \cos \theta/2$ and $OM_4 = 4r \cdot \cos \theta \cdot \cos^2 \theta/2$. The projection upon the y-axis gives the ordinate of the point, which is represented at P. A shift of the y-axis $2r$ units to the right ($r > 0$) completes the location of the curve.

This is a brief sketch of an effort to make the problem of curve tracing more a geometric than an arithmetic one. An acquaintance with the plans here indicated leads to a great variety of applications that serve to interest the student.

[1] Cf. Smith and Gale, *New Analytic Geometry*, p. 206, where this curve is drawn from table.

II. A Figure of Solid Analytic Geometry.

By John W. Bradshaw, University of Michigan.

There may well be a difference of opinion as to the particular system of axonometry that should be used in the figures of solid analytic geometry. Some would prefer drawings in perspective, as giving the best likeness and therefore furnishing the greatest assistance to the student's geometrical imagination; others would choose the simplest form of oblique parallel projection, that in which the picture-plane is assumed parallel to one of the coördinate planes, as lending itself most readily to reproduction by the student. The former type of figure is represented chiefly in the resort to photography of models,—Is there here a confession of inadequacy on the part of the draftsman?—the latter type is that commonly found in the texts. Perhaps a judicious use of both types with a comparison of the two would render the greatest service to the student.

Whatever attitude be taken on this question, however, it seems hardly necessary to contend that when a decision is once reached each figure should be consistently and accurately drawn according to the system chosen. Nor is much argument needed to show that definiteness is desirable, such definiteness as is attained by marking on the pictures of the coördinate axes three segments to represent three equal segments, or units, on the axes in space. Unless some such indication is given, one has no means of determining the point, or the direction, from which the figure should be viewed, a matter of the utmost importance if it is to produce the best possible impression. The drawings in most English and American texts with which I am acquainted are little short of disgraceful in their careless disregard of truth.

Let me call attention to the defects of just one simple figure as it appears in many books, the figure that accompanies the derivation of the equation of a plane in the normal form. Figure 1, but for the dotted lines and the lettering, is copied from a well-known text. To facilitate the description and distinguish relations in space from those of the plane figure, I shall designate points in space by Roman capitals and their pictures by the same letters starred.

In Figure 1, then, O^*X^*, O^*Y^*, and O^*Z^* represent three mutually perpendicular axes in space, OX, OY, and OZ. A plane η cuts these axes in U, V, W, represented by U^*, V^*, W^*. A perpendicular is let fall upon this plane from the origin, N is the foot of this perpendicular, O^*N^* is its picture. P is any point of the plane, reached from the origin by the broken line $OMLP$, whose parts OM, ML, LP give the coördinates of P. Since O^*X^* and O^*Z^* are perpendicular to each other it seems a reasonable assumption that the author had in mind a picture-plane parallel to the plane XOZ, though this cannot be positively asserted since he does not indicate the units. The first criticism of the figure does not depend on this assumption, the second does.

The point P, as end of the broken line $OMLP$, is not a point of the plane η. The dotted lines O^*K^* and K^*W^* are drawn to show this. The parallels OZ and LP determine a plane which cuts the plane XOY in OK and the plane η

in KW. If P lies in η it must lie in KW; so that Q^*, and not P^*, should be taken to represent a point of the plane. To be sure P^* is the picture of some point of the plane but not the point whose coördinates are indicated.

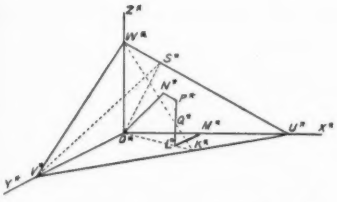

FIG. 1.

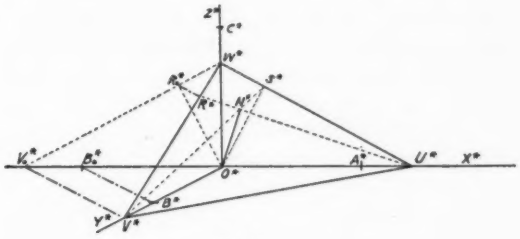

FIG. 2.

Again, N is an impossible position for the foot of the normal. For the plane ν of ON and OY is perpendicular both to XOZ and to η, and hence to their intersection UW. It cuts XOZ in a line OS perpendicular to UW. This must be represented by O^*S^* perpendicular to U^*W^*, since in this plane perpendiculars are represented by perpendiculars. Further, ν cuts η in VS and N must lie on this line, N^* on V^*S^*. The exact position of N^* cannot be given unless the unit on O^*Y^* be marked.

In Figure 2, O^*A^*, O^*B^*, O^*C^* represent equal segments on the axes in space, and the position of N is determined by rabatting the plane YOZ about the Z-axis till it coincides with XOZ. B^* thereby falls at B_0^*, such that $O^*B_0^* = O^*A^*$. The construction proceeds by drawing $V^*V_0^* \parallel B^*B_0^*$, $O^*R_0^* \perp V_0^*W^*$, $R_0^*R^* \parallel B_0^*B^*$. Then N^* is given as the intersection of U^*R^* and V^*S^*. We might instead have rabatted XOY about OX into coincidence with XOZ. The two constructions combined yield a useful check on the accuracy of the drawing.

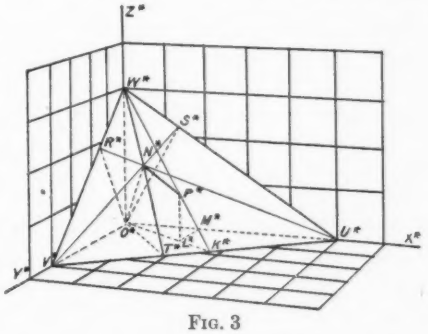

FIG. 3

As an illustration of what may be done with perspective Fig. 3 is offered. Would not many a student for whom Fig. 1 presents difficulties find Fig. 3 easily readable?

Has perhaps the author of the Solid Analytics been tricked into a careless and unwarranted sense of security by the generality of the theorem of Pohlke, that fundamental theorem of axonometry which asserts that any three segments of a plane, O^*X^*, O^*Y^*, O^*Z^* emanating from a point, so long as not more than three of the four points lie in a line, may be considered as the parallel projection of three equal, mutually perpendicular segments of space? Their lengths may

be taken at pleasure and the angles that they form with each other are entirely arbitrary. A superficial view of this theorem might lead one to think that the figures of axonometry may be drawn haphazard and that they will then represent the space figures in mind from some point of vision. On closer acquaintance, however, the theorem is seen to be associated with law, rather than license. It relates to setting up in the plane figure a standard of measurement which makes for definiteness.

RECENT PUBLICATIONS.

REVIEWS.

Girolamo Saccheri's Euclides Vindicatus. Edited and translated by GEORGE BRUCE HALSTED. Chicago and London, Open Court, 1920. 8vo. 30 + 246 pp. Price $2.00.

The Jesuit Girolamo Saccheri was born in 1667 and first taught in a college of his order in Milan where Tommaso Ceva, a brother of the Giovanni Ceva whose triangle theorem is well known, was teacher of mathematics. The influence of these brothers is apparent in the first two mathematical works which Saccheri published: (1) *Quæsita geometrica a Comite Rugerio de Vigintimilliis omnibus proposita, ab Hieronymo Saccherio Genuensi Societatis Jesu soluta.* Mediolani, 1693 (37 pages); another edition Parma, 1694 (dealing mainly with the solution of six problems in conic sections). (2) *Neostatica auctore Hieronymo Saccherio e Societate Jesu Excellentissimo senatui Mediolanensi dicata.* Mediolani, 1708 (168 pages) (discussing questions of statics and dynamics).

His third mathematical work *Euclides ab omni nævo vindicatus* was published at Milan in 1733 (16 + 142 pages + 6 plates). Saccheri died in October, 1733, and there is doubt as to whether he lived to see his completed master work issue from the press. While the work is frequently referred to during the next one hundred and fifty years it was not till the publication of an article by Beltrami[1] in 1889 that Saccheri became generally recognized as a forerunner of Legendre, Lobatchevsky, and Bolyai. Indeed Saccheri gave many of their propositions. For example, in his discussions of the parallel postulate, 1794–1833, Legendre proved, by using only the first twenty-eight propositions of Euclid's Elements, that: The sum of the angles of a triangle cannot be greater than two right angles; and that the sum must be equal to two right angles if this is true for a single triangle. Both of these propositions are proved in more general form by Saccheri.

In Saccheri's time the conception of parallels as equidistant straight lines was a favorite one, but Saccheri, like some of his predecessors, as Sommerville remarks, "saw that it would not do to assume this in the definition. He starts with two equal perpendiculars AC and BD to a line AB. When the ends C, D

[1] "Un precursore italiano di Legendre et di Lobatschewsky," *Atti della Reale Accademia dei Lincei*, Anno 1889, series 4, Vol. 5, pp. 441–448.

are joined, it is easily proved that the angles at C and D are equal; but are they right angles? Saccheri keeps an open mind, and proposes three hypotheses: (1) The hypothesis of the right angle; (2) The hypothesis of the obtuse angle; and (3) The hypothesis of the acute angle. The object of his work is to demolish the last two hypotheses and leave the first, the Euclidean hypothesis, supreme. . . . If Saccheri had had a little more imagination and been less bound down by tradition and by a firmly planted belief that Euclid's hypothesis was the only true one, he would have anticipated by a century the discovery of the two non-euclidean geometries which follow from his hypotheses of the obtuse and the acute angles."

Saccheri's work is divided into two books: the first, pages 1–101, containing the discussion indicated above; the second, a defense of the treatment of proportion found in book V of Euclid's Elements. The first complete translation of the first book was into German by Engel and Stäckel in their *Die Theorie der Parallellinien von Euklid bis auf Gauss*, Leipzig, 1895. This contains very full references to the literature of the work. An Italian translation, by G. Boccardini of both books (the first very slightly, the latter much abbreviated), appeared in the Manuali Hoepli series in 1904.

As to Dr. Halsted's translation there is no indication in the work before us that any part of the work has appeared in print elsewhere before. Nevertheless this is the case. The last paragraph of a circular advertising the book under review, and signed by Dr. Halsted, is as follows: "The English translation is a revision of the first ever made into any language, published in 1894, but long unprocurable." Hardly a single statement in this sentence is accurate. The translation in question (of propositions I–XXXVI in the first book) appeared in this MONTHLY, volumes 1–5, June, 1894–December, 1898. In "1894" the English translation of thirteen propositions only had appeared. The complete German translation of the first book was published before half of the English translation had appeared. The early numbers of the MONTHLY are not "unprocurable."

The differences between the part of Saccheri's work which appeared in this MONTHLY, and its reprint, are numerous—nearly three hundred were noticed in the first twenty-five propositions—but these have rarely introduced radical changes in the sense of the original passages. The figures are all new and are black on white instead of white on a black background. They are not repeated nearly as often as in the translation of Engel and Stäckel and the reading is consequently much less easy. In Saccheri's work the figures were on folding plates.

The additions to the original English translation are considerable: propositions 37 to 39 (the last of book 1), Saccheri's preface to the reader and synopsis of contents, and the original Latin of the whole, each page of text having the translation on the opposite page.

Dr. Halsted's "Introduction" emphasizes the importance of Saccheri's *Logica demonstrativa* first published in 1697 (the second edition appearing in 1701

and the third in 1735) in studying the *Euclides Vindicatus*. He has also appended two pages of notes, a subject index, and an index of proper names. The whole has been issued by the Open Court Publishing Company in a volume of attractive appearance.

This enterprising company and Dr. Halsted have placed mathematicians much in their debt by making the masterpieces of Saccheri,[1] Bolyai, and Lobatchevsky so readily accessible to American and English readers.

R. C. ARCHIBALD.

The Elementary Differential Geometry of Plane Curves. By R. H. FOWLER. (Cambridge Tracts in Mathematics and Mathematical Physics, No. 20.) Cambridge, at the University Press, 1920. 7 + 105 pages. Price 6 shillings.

Preface: "This tract is intended to present a precise account of the elementary differential properties of plane curves. The matter contained is in no sense new, but a suitable connected treatment in the English language has not been available.

"As a result, a number of interesting misconceptions are current in English text books. It is sufficient to mention two somewhat striking examples. (*a*) According to the ordinary definition of an envelope, as the locus of the limits of points of intersection of neighbouring curves, a curve is not the envelope of its circles of curvature, for neighbouring circles of curvature do not intersect. (*b*) The definitions of an asymptote—(1) a straight line, the distance from which of a point on the curve tends to zero as the point tends to infinity; (2) the limit of a tangent to the curve, whose point of contact tends to infinity—are not equivalent. The curve may have an asymptote according to the former definition and the tangent may exist at every point, but have no limit as its point of contact tends to infinity.

"The subjects dealt with, and the general method of treatment, are similar to those of the usual chapters on geometry in any *Cours d'Analyse*, except that in general plane curves alone are considered. At the same time extensions to three dimensions are made in a somewhat arbitrary selection of places, where the extension is immediate, and forms a natural commentary on the two dimensional work, or presents special points of interest (Frenet's formulæ). To make such extensions systematically would make the tract too long. The subject matter being wholly classical, no attempt has been made to give full references to sources of information; the reader however is referred at most stages to the analogous treatment of the subject in the *Cours* or *Traité d'analyse* of de la Vallée Poussin, Goursat, Jordan or Picard, works to which the author is much indebted.

"In general the functions, which define the curves under consideration, are (as usual) assumed to have as many continuous differential coefficients as may be mentioned. In places, however, more particularly at the beginning, this rule is deliberately departed from, and the greatest generality is sought for in the enunciation of any theorem. The determination of the *necessary and sufficient* conditions for the truth of any theorem is then the primary consideration. In the proofs of the elementary theorems, where this procedure is adopted, it is believed that this treatment will be found little more laborious than any rigorous treatment, and that it provides a connecting link between Analysis and more complicated geometrical theorems, in which insistence on the precise necessary conditions becomes tedious and out of place, and suitable sufficient conditions can always be tacitly assumed. At an earlier stage the more precise formulation of conditions may be regarded as (1) an important grounding for the student of Geometry, and (2) useful practice for the student of Analysis.

"The introductory chapter is a collection of somewhat disconnected theorems which are

[1] For other discussions of Saccheri's *Euclides Vindicatus* the curious reader may turn to Cantor, *Vorlesungen über Geschichte der Mathematik*, Vol. 3, 2te Aufl., 1901; to P. Mansion, "Analyse des recherches du P. Saccheri, S. J., sur le postulatum d'Euclide," *Annales de la Société scientifique de Bruxelles*, 1889–1890, Vol. 14, 2d part; reprinted in a supplement to *Mathesis*, January, 1891; to G. Veronese, *Grundzüge der Geometrie*, Leipzig, 1894, pp. 636–639; and to H. S. Carslaw's English translation of Bonola's *Non-Euclidean Geometry* (Open Court, 1912).

It seems curious that there is no reference to Saccheri in *The Catholic Encyclopedia*.

required for reference. The reader can omit it, and refer to it as it becomes necessary for the understanding of later chapters."

Contents—Chapter I: Introduction, 1–8; II: The elementary properties of tangents and normals, 8–23; III: The curvature of plane curves, 24–44; IV: The theory of contact, 45–58; V: The theory of envelopes, 58–79; VI: Singular points of plane curves, 80–89; VII: Asymptotes of plane curves, 89–103.

NOTES.

The second edition of W. A. Robertson and F. A. Ross's *Actuarial Theory* (Edinburgh, Oliver and Boyd, 23 + 431 pages) appeared in the early summer of 1920. The changes in the first edition are mainly in the correction of errors.

The Oxford University Press announces the following books: *Roger Bacon and the State of Science in the Fourteenth Century* by R. Steele; *Archimedes' Principle of the Balance and some criticisms upon it* by J. M. Child; *A History of Greek Mathematics*, 2 volumes, by Sir Thomas Heath—The Cambridge University Press has published a third edition of E. T. Whittaker and G. N. Watson's *A Course of Modern Analysis* (8 + 608 pages; price 40 shillings). It contains about 230 pages more than the first edition by Whittaker alone.

The second part of the sixth volume of the *Encyklopädie der Mathematischen Wissenschaften* is devoted to Astronomy. Between 1905 and 1915 six Hefte were published; the last of these included Professor E. W. Brown's "Theorie des Erdmondes," translated by A. v. Brunn. The first thirty-five pages of Heft 7, published in 1920, contain "Die Satelliten" by Professor Kurt Laves, of the University of Chicago. This section was "abgeschlossen im Sommer 1916." The rest of the Heft (pages 843–895) is devoted to "Bestimmung und Zusammenhang der astronomischen Konstanten" by J. Bauschinger.

In *Proceedings of the Benares Mathematical Society*, volume 2, 1920, D. K. Sen, research scholar in mathematics at Benares Hindu University, has an eleven page article entitled: "On the application of Burgess's method for determining the uniform motion of an ellipsoid of revolution through a viscous liquid along its axis of revolution." The reference here is to Professor R. W. Burgess's doctor's thesis at Cornell, on "The uniform motion of a sphere through a viscous liquid," published in the *American Journal of Mathematics*, 1916.

America's influence is being wielded in mathematics of the elementary and secondary schools of Cuba and South America through the works in Spanish by "Jorge Wentworth y David Eugenio Smith" published by the enterprising firm of Ginn and Company. These works are: *Geometría Plana y del de Espacio* (1915, 8 + 469 pp.; price \$1.72; translation of the authors' English work), *Elementos de Algebra* (1917, 6 + 458 pp.; price \$1.72; translation, with slight changes, of book 1 and part of book 2 of the authors' *School Algebra*), and *Aritmética Moderna*, 2 books (1916, 6 + 265 + 6 + 317 pp.; price \$.64 + \$.80; not an exact translation of any American editions).

From an editorial in *The New York Times*, October 3, 1920, on James Russell Lowell as a teacher at Harvard, with special reference to an article on this topic by W. R. Thayer: "Mr. Thayer describes and seems to wonder at Mr. Lowell's illimitable mustaches—frowning over a square beard. They were a work of art, nor can we understand why Mr. Thayer found them 'neither becoming nor beautiful.' There were barbate giants in those days. One remembers Professor BENJAMIN PEIRCE and his son, Professor JAMES MILLS PEIRCE, bearded both like the pard or the ball players of the House of David. Mr. James T. Fields, one of Mr. Lowell's publishers and his successor as editor of *The Atlantic*, had as copious and melodious a beard as ever inflamed a barber's eyes; at one time there were strange sausages or curls each side of the cheek."

The Göttingen *Nachrichten*, Mathematisch-physikalische Klasse for June, 1920, contains the first of a series of memoirs by G. H. Hardy and J. E. Littlewood in which they propose to develop in detail the new analytic method which they found for the discussion of Waring's problem and a number of allied problems in 'additiver Zahlentheorie.' The general lines of this method, in so far as it concerns Waring's problem in particular were explained in a recent paper in the *Quarterly Journal of Mathematics* [see this MONTHLY, *1920*, 272] where full references to the literature of the problem were given. The object of the authors now is "to give full details of the proofs, up to the point at which Hilbert's famous theorem, first proved in this journal in 1909, emerges as a corollary from our analysis." The Geschäftliche Mitteilungen of the *Nachrichten*, for July, 1920, contains sketches of the life and work of Woldemar Voigt [*1920*, 280], pages 45–52, by C. Runge, and of Adolf Hurwitz [*1920*, 191], pages 75–83, by D. Hilbert.

ARTICLES IN CURRENT PERIODICALS.

AMERICAN JOURNAL OF MATHEMATICS, volume 42, no. 3, July (published in October), 1920: "The failure of the Clifford chain" by W. B. Carver, 137–167; "On the representations of numbers as sums of 3, 5, 7, 9, 11 and 13 squares" by E. T. Bell, 168–188; "On a certain class of rational ruled surfaces" by A. Emch, 189–210.

BULLETIN DES SCIENCES MATHÉMATIQUES, volume 55, June, 1920: Review by R. Garnier of O. Veblen and J. W. Young's *Projective Geometry*, volume 2 (Boston, 1918), 105–112 [Last paragraph: "Par ce rapide aperçu on aura pu pressentir la variété des problèmes abordés, la rigueur des développements et l'importance des résultats établis. A ces qualités essentielles, le Traité joint tous les avantages d'une forme attrayante; rédigé dans un style clair et précis, complèté par un choix très étendu d' 'exercises'—dont quelques-uns, d'un niveau élevé, sont empruntés à de recents Mémoires—, l'Ouvrage de MM. Veblen et Young sera pour l'étudiant français le plus sûr des initiateurs."]

BULLETIN OF THE AMERICAN MATHEMATICAL SOCIETY, volume 27, no. 1, October, 1920: "The Seattle meeting of the San Francisco section" by E. T. Bell, 1–4; "Note on a generalization of a theorem of Baire" by E. W. Chittenden, 5–6; "Certain iterative characteristics of bilinear operations" by N. Wiener, 6–10; "Necessary and sufficient conditions that a linear transformation be completely continuous" by C. A. Fischer, 10–17; "On the relation of the roots and poles of a rational function to the roots of its derivative" by B. Z. Linfield, 17–21; "Moritz Cantor, the historian of mathematics" by F. Cajori, 21–28; "Shorter notices," 28–38 [Reviews by D. E. Smith of A. Eymieu's *La Part des Croyants dans les Progrès de la Science au XIX^e Siècle*, 3e éd. (Paris, 1920) and of J. M. Child's *Early Mathematical Manuscripts of Leibnig* (Chicago, 1920); reviews by R. D. Carmichael of A. N. Whitehead's *An Enquiry Concerning the Principles of Natural Knowledge* (Cambridge, 1919) and of H. Bateman's *Differential Equations* (London, 1918)]; "Notes," 39–46; "New publications," 46–48.

MATHEMATICAL GAZETTE, volume 10, July, 1920: "The Leeds meeting" by R. C. Fawdry, 81–82; "The training of the mathematical teacher" by W. P. Milne, 83–85; "The mathematics of thread and cloth construction: an historical survey" by A. F. Barker, 86–91; "The mathematical theory of the sateen arrangement" by S. A. Shorter, 92–97; "The teaching of mathematics to textile students" by J. H. Whitwam, 98–102; "On the teaching and applications of dynamics" by A. Gray, 103–114; "Mathematics and commerce" by A. N. Shimmin, 115–118; "Life of James Stirling, the Venetian" by C. Tweedie, 119–128.

MESSENGER OF MATHEMATICS, volume 49, nos. 7, 8, November, December, 1919: "The tetrahedron and pentaspherical coördinates" by T. C. Lewis, 97–106; "Two trigonometrical determinants" by E. H. Neville, 107–111; "The dissection of rectilineal figures" (concluded) by W. H. Macaulay, 111–121; "On Laplace's integrals for a Legendre polynomial" by S. Pollard, 121–125; "A form for $(d/dn)P_n(\mu)$, where $P_n(\mu)$ is the Legendre polynomial of degree n" by A. E. Jolliffe, 125–127; "On a property of algebraic numbers" by W. Burnside, 127–128—January, February, March, 1920: "On plane curves of degree n with tangents of n-point contact" by H. Hilton, 129–134; "An integral equation occurring in a mathematical theory of retail trade" by H. Bateman, 134–137 [First two paragraphs: "A tradesman, who buys and sells various articles, will be supposed to have worked up his business to such an extent that he can be sure of selling his goods at a constant rate so that a new supply of any article will be completely exhausted at the end of an interval of time T after the date of purchase.[1] Our problem is to find the law according to which goods must be purchased in order that the total value of the stock may remain constant.

"To simplify matters the process of buying and selling will be treated as continuous instead of discontinuous. This is approximately true if business is brisk all the time and if the working hours and days are pieced together so that 'business time' can be treated as a continuous variable."]; "On apolar and co-apolar triangles for a cubic, and on apolarly conjugate triangles" by E. B. Elliott, 137–149; "Notes on some points in the integral calculus" (liii) by G. H. Hardy, 149–155; "Four-vector algebra and analysis" by C. E. Weatherburn, 155–176—April: "On the congruence $(p-1)! \equiv -1 \pmod{p^2}$" by N. G. W. H. Beeger, 177–178; "On Pascalian collinearities and concurrencies" by E. B. Elliott, 178–180; "Transitive constituents of the group of isomorphisms of any abelian group of order p^m" by G. A. Miller, 180–186; "Is there an analogue in solid geometry to Feuerbach's theorem ?" by T. C. Lewis, 187–192 [A discussion of a problem proposed in J. L. Coolidge, *Treatise on the Circle and the Sphere*, p. 247; Mr. Lewis finds: "It is easy to prove that there is no analogue, in space of three dimensions, to Feuerbach's theorem, and it follows that there is nothing corresponding to the Hart systems"].

THE MONIST, volume 30, no. 3, July, 1920: "Space and time" by R. W. Sellars, 321–364; "Lord Rayleigh," 474–475.

NATURE, volume 105, July 29, 1920: "Tycho Brahe" by J. K. Fotheringham, 672–673 [Review of *Tychonis Brahe Dani Opera Omnia*, volume 6, Copenhagen, 1919].—August 5: "Relativity and reality" by R. A. Sampson, 708.—August 12: "Complex elements in geometry" by G. B. Mathews, 736–737 ["P.S.—Since the above was written, I have had time to reflect further upon Prof. Hatton's book, and I have read Prof. G. H. Hardy's review of it in a recent number of the *Mathematical Gazette*. I do not wholly agree with Prof. Hardy's attitude, because I still think that there are geometrical notions not reducible to arithmetic—still less to formal logic. But I do agree with him that Prof. Hatton's book has no theoretical value, and disagreeable as it is, I think it is my duty to say so, especially as I have been informed that another reviewer has praised the book in absurdly exaggerated terms. G.B.M." Cf. *1920*, 268–269]; "Obituary, Professor John Perry, F.R.S.," by H.E.A. and W.E.D., 751–753; [Note], 762 ["News has just reached us that Prof. A. T. De LURY was appointed some months ago to be the head of the department of mathematics in the University of Toronto by the Board of Governors on the recommendation of the president of the University, Sir R. A. Falconer. The staff, council, and senate have nothing to do with appointments, and the only check upon the action of the president and the Board of Governors is public opinion. Prof. De Lury has been a member of the teaching staff of the University for many years, and is the author of a number of mathematical text-books which have done service in the schools of the province of Ontario. He possesses high teaching ability, but has not been associated with the research activities which it should be the essential

[1] "Strictly this should be the date at which the new supply is made available for sale. We shall suppose for simplicity that in the case of any special article the date at which the old supply is exhausted coincides approximately with the date when a new package is opened."

function of a university to create and foster. Without men engaged in the production of new knowledge the work of a university differs little from that of a secondary school preparing students for examinations. Toronto has won much distinction by the scientific investigations of such men as Profs. Maccallum, McLennan, and Brodie, and it was hoped that the chair of mathematics would have been filled by someone who possesses the highest research qualifications in mathematics that Canada could produce. If Prof. De Lury can and will build up a strong research staff under him, he will be doing the best service to his University and extend the stimulating atmosphere which some of his scientific colleagues have given to the institution by their work"].—August 19: "The mathematician as anatomist" by A. Keith, 767–770 [Review of two parts of K. Pearson's "A Study of the long bones of the English skeleton" Cambridge, 1919]; "Obituary, Sir Norman Lockyer," 781–784.—August 26: "Professor Alexander's Gifford Lectures" by Viscount Haldane, 798–801 [Review of S. Alexander's *Space, Time, and Deity*, 2 volumes, London, 1920]; "Use of Sumner lines in navigation" by J. Ball, 806–808; "Relativity and hyperbolic space" by A. McAulay, 808; "Sir Norman Lockyer's contributions to astrophysics," by A. Fowler, 831–833.—Volume 106, September 2: "Internal constitution of the stars" by A. S. Eddington, 14–20; "Memorial tributes to Sir Norman Lockyer," 20–25 [by various authors]—September 16: "Ewing's 'Thermodynamics'" by H.L.C., 72–73 [Review of J. A. Ewing's *Thermodynamics for Engineers*, Cambridge, 1920]; "Associated squares and derived squares of order 5" by J. C. Burnett, 79 [shows that there are nearly 700,000 such magic squares of the first 25 integers].

La Nature, volume 482, July 10, 1920: "La 'multi,' nouvelle machine à multiplier" by J. Boyer, 30–32—July 17: "L'heure exacte sans instruments et sans calculs" by Viguier, 43–45.

Nouvelles Annales de Mathématiques, volume 79, May, 1920: "Sur l'application de la loi de Gauss à la position probable d'un point dans le plan ou dans l'espace" (continued) by J. Haag, 161–178; "Sur une propriété caractéristique des cylindres et du cylindroïde" by R. Harmegnies, 178–180; "Sur la surface dont tous les points sont des ombilics" by R. Harmegnies, 180–181; "Courbes gauches liées par échange des directions des tangentes et des binormales. Les formules de Frenet sont intuitives" by G. Fontené, 181–188; "Correspondance," "Chronique," "Solutions de questions proposées," "Questions," 188–200—June: "Sur l'application de la loi de Gauss à la position probable d'un point dans le plan ou dans l'espace" (concluded) by J. Haag, 201–208; "Sur un système remarquable de cinq droites" by R. Bricard, 209–214; "Rayon de courbure de la courbe qui est le lieu des centres des sphères osculatrices à une courbe gauche" by G. Fontené, 214–219; "Chronique," "Certificats d'analyse supérieure," "Solutions de questions proposées," "Questions," 220–240—July: "Sur un défaut de la méthode d'interpolation par les polynomes de Lagrange" by M. Fréchet, 241–249; "Transformation polaire interaxiale" by M. D'Ocagne, 248–260; "Agrégation des sciences mathématiques (juillet, 1919): Problème de calcul différentiel et intégral, et composition de calcul différentiel et intégral" by R. Garnier, 260–275; "Chronique" and "Questions," 275–280—August: "Surfaces de translation applicables l'une sur l'autre" by B. Gambier, 281–295; "Notes sur les congruences de normales" by M. Bayard, 295–297; "Concours spécial d'agrégation de 1919. Solution du problème de mécanique" by E. Delassus, 297–307; "Certificat de géométrie supérieure, Paris, 1919," 307–311; "Certificat de mécanique appliquée, Lille, 1919," 314–316; "Concours d'admission a l'Ecole Polytechnique en 1920," 316–319; "Questions," 319–320.

Proceedings of the Royal Society of Edinburgh, volume 40, part 1, January, 1920: "Notices of fellows . . . recently deceased" [E. C. Fisher; Lord Rayleigh, J. W. Strutt; J. M. Bernard], 1–2.

Rendiconti del Circolo Matematico di Palermo, volume 44, no. 1, January–April, 1920: "The foundations of the elliptic functions" by H. Hancock, 87–102.

Revue de l'Enseignement des Sciences, volume 14, March–April, 1920: "Sur le sens de la variation d'une fonction" by G. Fontené, 49–53; "Sur l'ensemble de deux équations et la fraction rationnelle du second degré" by J. Juhel-Rénoy, 53–60; "Perpendiculaire menée d'un point sur une droite" by J. Lemaire, 60; "La fonction exponentielle et les fonctions circulaires" by C. Michel, 61–84; "Examens et concours de 1919," 84–91; "Problèmes de mathématiques donnés au baccalauréat en juillet 1919," 91–96—May–June, 1920: "Première leçon sur les séries entières" by P. Flamant, 97–107; "Note de géométrie" by R. Malloizel, 107; "Sur les orbiformes" by C. Bioche, 108–110; "Sur les ovales de Descartes" by R. Bérard, 110–119; "Sur le développement d'un cône et les points d'inflexion de la transformée d'une courbe tracée sur le cône" by R. Bérard, 119–121.

REVUE DE MATHÉMATIQUES SPÉCIALES, volume 13, no. 9, June, 1920: "Sur la folium double" by G. Loria, 201–202; Solutions of questions in analytic geometry, algebra and analysis, mechanics, 203–207, 213–220; "Sujets de concours, agrégation des sciences mathématiques, session spéciale d'octobre, 1919," 210–213.—No. 10, July: Solutions of questions in analytic geometry, mechanics, calculus, analysis, 225–233, 238–243; "Ecole Normale Supérieure et Bourses de Licence, concours de 1919," 234–237.—No. 11, August: "Ecole Normale Supérieure et Bourses de Licence, concours de 1914," 249–257 [Solutions of problems by L. Simon]; "Sujets de concours, Ecole Polytechnique, 1920," 258–260; Sujets de concours, Ecole Navale, 1916 et 1917, 268–272.

REVUE GÉNÉRALE DES SCIENCES, volume 31, June 30, 1920: "Charles Ange Laisant (1841–1920)" by J. Boyer, 397–398—August 15–30, 1920: "Les relations entre la science et l'industrie et les sociétés de perfectionnements industriels" by R. d'Adhémar, 513–519.

SCIENCE, new series, volume 52, July 23, 1920: "The structure of the universe" by W. D. MacMillan, 67–74—August 6: "A priori use of the Gaussian law" by E. G. Boring, 129–130—August 13: "Transverse vibrating rods" by A. G. Webster, 154; "Mathematische Zeitschrift" by G. A. Miller, 155—August 20: "Efficiency in thermal calculations" by A. W. Forbes and A. G. Webster, 175–176; Review by F. H. Garrison of C. Singer's *Greek Science and Modern Science* (Oxford Univ. Press, 1920), 178–179—September 3: Review by F. H. Garrison of D. W. Singer's *Hand-List of Scientific Manuscripts in the British Isles dating from before the sixteenth century* (London, 1919) and *Survey of Medical Manuscripts in the British Isles dating from before the sixteenth century* (London, 1920), 216–227—September 10: "The internal constitution of the stars" by A. S. Eddington, 233–240 [Address before the mathematical and physical section of the British Association for the Advancement of Science]—September 17: "Galileo's experiment from the leaning tower" by E. A. Partridge, 272–273—September 24: "The national committee on mathematical requirements," 289—October 1: "Electricity and gravitation" by H. Bateman, 314–315.

SCIENTIFIC MONTHLY, volume 11, September, 1920: "Giant suns" by H. H. Turner, 228–234; "A simplified musical notation" by E. V. Huntington with an introduction by A. T. Davison, 276–283 [First sentences of the article: "The purpose of this paper is to present a new musical notation which, while retaining all the excellent features of the present notation, would, it is believed, greatly simplify the processes of reading, studying and composing musical scores.

"The plan is not one of the artificial mnemonic devices of which the literature is full, but is based directly on the fundamental principle of music, namely, the equi-tempered scale. This scale, introduced by J. S. Bach about two centuries ago, and now dominating all musical composition, makes use of only twelve notes in each octave, these twelve notes dividing the octave into twelve mathematically equal intervals, called semi-tones. Now, if we draw an ordinary five-line staff with one ledger line, we see that exactly twelve notes can be accommodated on the lines and spaces of such a staff. What is then more natural than to assign one place on the staff to each of the twelve notes of the scale, thus doing away with all 'sharps' and 'flats,' and representing every musical interval correctly to the eye as well as to the ear? This, in brief, is precisely the plan here proposed."]

TRANSACTIONS OF THE AMERICAN MATHEMATICAL SOCIETY, volume 21, no. 3, July (published in October), 1920: "On the representation of a number as the sum of any number of squares, and in particular of five" by G. H. Hardy, 255–284; "A memoir upon formal invariancy with regard to binary modular transformations. Invariants of relativity" by O. E. Glenn, 285–312; "Properties of the subgroups of an abelian prime power group which are conjugate under its group of isomorphisms" by G. A. Miller, 313–320; "On the order of magnitude of the coefficients in trigonometric interpolation" by D. Jackson, 321–332; "Concerning simple continuous curves" by R. L. Moore, 333–347; "On the iteration of rational functions" by J. F. Ritt, 348–356.—October: "Minima of functions of lines" by Elizabeth Le Stourgeon, 357–383; "Invariants of infinite groups in the plane" by E. F. Simonds, 384–390; "On triply orthogonal congruences" by J. B. Shaw, 391–408; "A set of properties characteristic of a class of congruences connected with the theory of functions" by E. J. Wilczynski, 409–445; "On the equilibrium of a fluid mass at rest" by J. W. Alexander, 446–450; "Concerning approachability of simple closed and open curves" by J. R. Kline, 451–458.

ZEITSCHRIFT FÜR MATHEMATISCHEN UND NATURWISSENSCHAFTLICHEN UNTERRICHT, volume 51, no. 6, published June 15, 1920: "Zur Erfindung des Zeichens $\times$" by H. Wieleitner, 145–148; "Optische Geometrie" by R. Böger, 148–164; "Kleine Mitteilungen," 164–167; "Bücherbesprechungen," 167–169.

PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL AND OTTO DUNKEL.

Send all communications about Problems and Solutions to B. F. FINKEL, Springfield, Mo.

PROBLEMS FOR SOLUTION.

[N.B. The editorial work of this department would be greatly facilitated if, on sending in problems, the proposers would also enclose their solutions—*when they have them*. If a problem proposed is not original the proposer is requested *invariably* to state the fact and to give an exact reference to the source.]

2869. Proposed by the late L. G. WELD.

The successive segments of a broken right line are represented by the successive terms of the harmonic progression, $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}$, *ad infinitum*. Each segment makes with the preceding a given angle θ. What is the distance and what is the direction of the limiting point (if there be such) from the initial point of the first segment?

2870. Proposed by WARREN WEAVER, University of Wisconsin.

A pendulum bob of mass m is attached to one end of a weightless and inextensible string of length l and swings as a conical pendulum with an angular velocity ω_1 about a vertical line through a fixed point to which the other end of the string is attached. If the angular velocity is increased to ω_2, the height through which the bob rises is independent of the length l. Consider, then, a very long and a very short pendulum. Suppose they are each swung first with an angular velocity ω_1 and then with a larger angular velocity, ω_2, the difference between these two values being great enough so that the longer pendulum rises through a height greater than the length of the shorter pendulum. According to the above result the shorter one should rise through this same height, which is obviously impossible. Explain this apparent paradox.

2871. Proposed by the late L. G. WELD.

Weight being disregarded, a package may be admitted to the parcels post if the length plus the greatest girth, measured transversely to the length, does not exceed 72 inches. What is the size of the smallest square window through which all admissible rectangular boxes can be passed?

2872. Proposed by W. D. LAMBERT, U. S. Coast and Geodetic Survey.

The rectangular coördinates of a point P at the time t are given by the equations

$$x = k \cos \gamma \cos (nt - \alpha), \qquad y = k \sin \gamma \cos (nt - \beta),$$

where k, γ, n, α, and β are constants and γ is taken in the first quadrant. An auxiliary angle δ, in the first quadrant, is defined by the equation

$$\sin \delta = \sin 2\gamma \cdot |\sin (\alpha - \beta)|.$$

(1) Show that P describes an ellipse the lengths of whose semi-axes are $k \cos \frac{1}{2}\delta$ and $k \sin \frac{1}{2}\delta$ and the inclinations of whose axes to the x-axis are given by

$$\tan 2\theta = \tan 2\gamma \cos (\alpha - \beta).$$

(2) Show that the time T when P is at the end of an axis is given by

$$\tan (2nT - \alpha - \beta) = \cos 2\gamma \tan (\alpha - \beta).$$

(3) Obtain criteria for distinguishing between the major and minor axes and for the direction of rotation of P and show that the quantities 2γ, $\alpha - \beta$, δ, 2θ and $2nT - \alpha - \beta$ are simply related as the parts of a right spherical triangle, thus providing a partial check on the computation.

2873. Proposed by D. H. RICHERT, Bethel College, Newton, Kan.

At B is the enemy's battery. At M_1 a battery is to be placed to silence B. Listening posts are installed at M_1, M_2, M_3, all provided with stop-watches. From the maps at hand, the three sides of the triangle $M_1M_2M_3$ are known. B is not visible from any one of the points M_1, M_2, M_3.

The sound of a gun fired at B reaches M_1 at the time T, and M_2 at the time $T + \tau_1$ sec., and it reaches M_3 at the time $T + \tau_2$ sec. How far is B from M_1?

2874. Proposed by J. L. RILEY, Stephenville, Texas.

Show that the equation,

$$x^n + ax^{n-1} + bx^{n-2} + \cdots + k = 0,$$

has some imaginary roots if $a^2 - 2b < n\sqrt[n]{k^2}$; $a, b, \cdots k$ are supposed real.

Note. This result is a particular case of a theorem contained in a paper published a few years ago.—EDITORS.

2875.

Show how to draw through a given point a straight line dividing a given triangle into two parts of equal area.

Three cases of this well-known problem (proposed by an Association member) were discussed by Euclid over two thousand years ago. A new solution is sought. —EDITORS.

PROBLEMS—NOTES

1. Mr. R. M. Winger, of the University of Washington, suggested the problem: "Six of the points of the Feuerbach circle of a triangle, namely, the middle points of the sides of the triangle and the middle points of the junctions of the opposite vertices with the orthocenter, lie in pairs at the extremities of diameters." This result is the basis of several proofs that Feuerbach's circle goes through the six points noted above as well as through the feet of the altitudes. See, for example, J. Casey, *Sequel to . . . the Elements of Euclid*, third edition, 1884, p. 58.

H. P. M.

2. The following problem was proposed and solved in the *Journal of the Indian Mathematical Club*, 1910: "All the significant figures except four in a sum of division were replaced by dots, and the result was:

$$\begin{array}{r} \cdots)\cdots\cdots(\cdots \\ \underline{\cdot 0 \cdot \cdot} \\ \cdots \\ \underline{\cdot 50 \cdot} \\ \cdots \\ \underline{\cdot 4 \cdot} \end{array}$$

It is required to recover the lost figures."

Similar problems with "four fours" and "seven sevens" are due to Mr. W. E. H. Berwick, lecturer in University College, Bangor, England:

$$\begin{array}{r} \cdots)\cdots\cdots 4(\cdot 4 \cdot \cdot \\ \underline{\cdots} \\ \cdot \cdot 4 \cdot \\ \underline{\cdots\cdot} \\ \cdots\cdot \\ \underline{\cdot 4 \cdot} \\ \cdots\cdot \\ \underline{\cdots\cdot} \end{array} \qquad \begin{array}{r} \cdots\cdot 7 \cdot)\cdot\cdot 7 \cdots\cdots(\cdot\cdot 7 \cdot\cdot \\ \underline{\cdots\cdots} \\ \cdots\cdot 7 \cdot \\ \underline{\cdots\cdots} \\ \cdot 7 \cdots\cdot \\ \underline{\cdot 7 \cdots\cdot} \\ \cdots\cdots \\ \underline{\cdots 7 \cdot \cdot} \\ \cdots\cdots \\ \underline{\cdots\cdots} \end{array}$$

The first of these was proposed in *The Mathematical Gazette*, March, 1920, page 43, with the statement: "There are just four ways of filling up the missing figures so as to leave a correctly-worked long division sum (in scale ten)." The problem was reproduced in *The Observatory*, July, 1920, page 274, and the solutions were given on page 372 of the issue for October. The "seven sevens" problem was proposed in *The School World*, London, July 1906, volume 8, page 280; its solution appeared in the following August number, page 320. The proposer remarked that "there is one, and only one solution possible." ARC.

3. A member of the Association has proposed as a problem the derivation of the familiar formula:

$$\cos nx = 2^{n-1}\cos^n x - 2^{n-3}\,n\cos^{n-2}x + 2^{n-5}\frac{n(n-3)}{2\,!}\cos^{n-4}x - \cdots$$

$$+ (-1)^r\frac{2^{n-2r-1}n(n-r-1)(n-r-2)\cdots(n-2r+1)}{r\,!}\cos^{n-2r}x + \cdots$$

where n is a positive integer. Such a derivation may be found in many well known works, for example: R. Levett and C. Davison, *Elements of Plane Trigonometry*, London, 1892, pp. 299–300; S. L. Loney, *Plane Trigonometry*, Cambridge, 1893, pp. 323–354; J. A. Serret, *Traité de Trigonométrie*, 6e édition, Paris, 1880, pp. 204–236; T. J. I'A. Bromwich, *An Introduction to the Theory of Infinite Series*, London, 1908, p. 177; and H. Weber and J. Wellstein, *Encyklopädie der Elementar-Mathematik*, Band 2, 2te Auflage, Leipzig, 1907, pp. 320–321. In obtaining the result, DeMoivre's theorem, the expansion of $\log(1-2p\cos x+p^2)$,[1] and mathematical induction are variously employed (cf. problem 2828, *1920*, 186).

References may be given also to discussions in Lagrange, *Leçons sur le Calcul des Fonctions*, seconde édition, Paris, 1806; in *Cambridge Mathematical Journal*, vol. 4, November, 1844, pp. 250–252, by R. Moon; and in *Nouvelles Annales de Mathématiques*, 2e série, vol. 12, 1873, pp. 408, 425, by Mourgue and LeBesgue; vol. 14, 1875, p. 385, by Desboves.

In a letter to Leibnitz dated June 16, 1676, Newton gave a formula which, in modern notation, may be written:

$$\sin nx = n\sin x + \frac{(1-n^2)n}{3\,!}\sin^3 x + \frac{(1-n^2)(9-n^2)n}{5\,!}\sin^5 x + \cdots.$$

About twenty-five years later Johann Bernoulli derived the formula which, in modern notation, may be written:

$$\cos nx = \cos^n x - \binom{n}{2}\sin^2 x\cos^{n-2}x + \binom{n}{4}\sin^4 x\cos^{n-4}x - \cdots.$$

The formula with which this note opens seems to have been first given in 1748 by Euler, who arrived at the result by mathematical induction. (See *Introductio in analysin infinitorum*. Tomus primus. Lausannae, 1748; French edition,

[1] This method was used by E. Heine, *Mathematische Annalen*, vol. 2, 1870, pp. 187–188.

Paris, 1796, pp. 194–195. Also "Dilucidationes super formulis, quibus sinus et cosinus angulorum multiplorum exprimi solent, ubi simul ingentes difficultates diluuntur," *Nova acta acad. sc. Petrop.*, volume 9 (1791), 1795, pp. 54–55; "Conventui exhib. die 6 Mart. 1777.") In a posthumous paper, "Enodatio insignis cujusdam paradoxi circa multiplicationem angulorum observati," (*Opera postuma*, vol. 1, 1862, pp. 299–314) Euler discussed cos $n\phi$ as a power series in cos ϕ.

The developments of sin nx and cos nx into series in sin x and cos x, when n is not restricted to integral values, seem to have been first thoroughly discussed by Poinsot and forms the subject of his *Recherches sur l'Analyse des Sections Angulaires*, Paris, 1825, 80 quarto pages. ARC.

SOLUTIONS OF PROBLEMS.

417 (Algebra) [1914, 156]. Proposed by A. J. RICHARDSON, Marquette, Mich.

Required to reduce the quartic

$$x^4 + px^2 + qx + r = 0$$

to the form

$$(x^2 + k)^2 = [(2k - p)x + (2k - q)]^2,$$

wherein k is the solution of a certain cubic.[1] Hence, express the solution of the given quartic in terms of p, q, r, and k.

SOLUTION BY H. S. UHLER, Yale University.

We shall show that the hypothesis that the given quartic can be reduced to the required form destroys the mutual independence of the coefficients p, q, r, and that k is not "the solution of a certain cubic."

Transposing the right-hand member of

$$(x^2 + k)^2 = [(2k - p)x + (2k - q)]^2,$$

expanding, and collecting terms, we find

$$x^4 + [2k - (2k - p)^2]x^2 - 2(2k - p)(2k - q)x + [k^2 - (2k - q)^2] = 0.$$

Since, by hypothesis, the last equation is a modified form of the given quartic the coefficients of like powers of x in the two equations must be equal, that is, the three following conditions must hold

$$2k - (2k - p)^2 = p, \tag{1}$$

$$-2(2k - p)(2k - q) = q, \tag{2}$$

$$k^2 - (2k - q)^2 = r. \tag{3}$$

Condition (1) gives

$$2k - p = 0, \tag{4}$$

or

$$2k - p = 1. \tag{5}$$

In either case k is a linear function of p alone and hence k is not dependent upon the solution of any cubic.

Combining relations (2) and (4) we see that q must have the special value zero. Then the given quartic reduces to a quadratic in x^2 and the solution is immediate.

[1] There must have been some mistake in writing this problem. The following may have been intended:

$$(x^2 + k)^2 = [x\sqrt{2k - p} - \sqrt{k^2 - r}]^2.$$

See Todhunter, *An Elementary Treatise on the Theory of Equations*, London, 1880, p. 117.—EDITORS.

Combining relations (2) and (5) we obtain

$$k = q/4. \tag{6}$$

Equating the values of k given by relations (5) and (6) we find that p and q are connected by the linear equation

$$2p - q + 2 = 0. \tag{7}$$

Substituting k from relation (6) in relation (3) and then employing equation (7) we find

$$r = -3q^2/16 = -3(p+1)^2/4. \tag{8}$$

Incidentally, the last result shows that r must be negative in order that p and q may be real. Relations (7) and (8) enable the given quartic to be expressed in terms of (say) p alone, namely

$$x^4 + px^2 + 2(p+1)x - 3(p+1)^2/4 = 0,$$

or

$$[x^2 + (p+1)/2]^2 = [x - (p+1)]^2.$$

Consequently, the roots of the quartic are

$$x = (1 \pm \sqrt{-6p-5})/2,$$

$$x = (-1 \pm \sqrt{2p+3})/2.$$

For $-3/2 \leqq p \leqq -5/6$ all four roots will be real. For values of p outside of this range two roots will be real and two complex.

The reduction of the general quartic to the difference of two squares is given in Burnside and Panton's *Theory of Equations*, vol. I, p. 129, Art. 63 (4th or 7th editions.)

2783 [1919, 312]. Proposed by C. C. BRAMBLE, U. S. Naval Academy.

Two players, A and B, take turns throwing a single dice, A leading. The one first making a score of three aces is to be the winner. Find the probability that A will win.

SOLUTION BY H. P. MANNING, Brown University.

The probability of throwing three aces in the first three throws is $1/6^3$; in general, the probability of throwing exactly two aces in the first n throws, and an ace in the $(n+1)$th throw, is $\frac{n(n-1)}{2} \cdot \frac{5^{n-2}}{6^{n+1}}$. That is, the probabilities of the different ways of throwing the first three aces are for A or B

$$\frac{1}{6^3}, \quad 3\frac{5}{6^4}, \quad 6\frac{5^2}{6^5}, \quad \cdots.$$

Call these numbers

$$a \quad b \quad c \quad \cdots.$$

Their sum should equal 1.

The probability that A wins is

$$a + (1-a)b + (1-a-b)c + \cdots,$$

the factor in the parenthesis in any term indicating the probability that B has not yet thrown three aces. This expression may be written

$$a + b + c + \cdots$$
$$-(ab + ac + bc + \cdots).$$

Similarly the probability that B wins is

$$(1-a)a + (1-a-b)b + \cdots,$$

which may be written

$$a + b + c + \cdots$$
$$-(a^2 + ab + b^2 + ac + bc + c^2 + \cdots).$$

Put

$$M = a + b + c + \cdots,$$
$$N = a^2 + b^2 + c^2 + \cdots$$
$$P = ab + ac + bc + \cdots,$$

$$M = \frac{1}{2\cdot 6^3}\left[1\cdot 2 + 2\cdot 3\cdot\frac{5}{6} + 3\cdot 4\left(\frac{5}{6}\right)^2 + \cdots\right],$$

$$N = \frac{1}{4\cdot 36^3}\left[(1\cdot 2)^2 + (2\cdot 3)^2\frac{25}{36} + (3\cdot 4)^2\left(\frac{25}{36}\right)^2 + \cdots\right].$$

Let

$$f(x) = 1 + x + x^2 + \cdots = \frac{1}{1-x} \quad (|x| < 1);$$

then

$$f''(x) = 1\cdot 2 + 2\cdot 3x + 3\cdot 4x^2 + \cdots = \frac{2}{(1-x)^3}.$$

Hence putting $x = \frac{5}{6}$ we have $M = \frac{1}{2\cdot 6^3}\cdot\frac{2}{(1/6)^3} = 1.$

$$x^2f''(x) = 1\cdot 2x^2 + 2\cdot 3x^3 + 3\cdot 4x^4 + \cdots.$$

Differentiating twice,

$$2f''(x) + 4xf'''(x) + x^2f^{iv}(x) = (1\cdot 2)^2 + (2\cdot 3)^2x + (3\cdot 4)^2x^2 + \cdots.$$

This equals

$$2\frac{2}{(1-x)^3} + 4x\frac{6}{(1-x)^4} + x^2\frac{24}{(1-x)^5} = \frac{4}{(1-x)^5}(1 + 4x + x^2).$$

Hence putting $x = \frac{25}{36}$ we have

$$N = \frac{1}{4\cdot 36^3}\cdot\frac{4\cdot 36^5}{11^5}\left(1 + 4\frac{25}{36} + \frac{25^2}{36^2}\right) = \frac{36^2 + 4\cdot 25\cdot 36 + 25^2}{11^5} = \frac{5521}{11^5}.$$

$$M^2 = N + 2P = 1;$$

hence

$$P = \frac{1}{2} - \frac{5521}{2\cdot 11^5}.$$

Now the probability that A wins is $1 - P$ and the probability that B wins is $1 - N - P$. and so we may say, the probability that A wins is $N + P$ and the probability that B wins is P; These probabilities are

for A $\quad \frac{1}{2} + \frac{5521}{2\cdot 11^5}$,

" B $\quad \frac{1}{2} - \frac{5521}{2\cdot 11^5}$.

All of these series are absolutely convergent power series, the parentheses may be removed and the terms rearranged as shown in the expressions for the two probabilities, and M may be squared to give the relation $M^2 = N + 2P$.

NOTES AND NEWS.

It is hoped that readers of the MONTHLY will cooperate in contributing to the general interest of this department by sending items to the Editor-in-Chief.

Miss JESSIE M. SHORT, formerly of Carleton College but recently with the National Workmen's Compensation Service Bureau of New York, has been appointed instructor in mathematics at Reed College.

R. A. WELLS, dean and professor of mathematics at Park College, Missouri, has been appointed associate professor of mathematics in the Michigan State Normal College.

Dr. H. F. MACNEISH, instructor in mathematics at the College of the City of New York, has been promoted to an assistant professorship.

Mr. B. D. ROBERTS has been appointed professor of mathematics at Parsons College, Fairfield, Iowa.

Instructor C. D. EHRMAN, of the University of Wisconsin (*1920*, 88), has been appointed assistant professor of mathematics at Richmond College.

Mr. A. S. ADAMS, Mr. T. H. JOHNSON, and Mr. W. L. LUCAS have been appointed instructors in mathematics at the University of Maine.

Professor I. L. MILLER, of Carthage College, Illinois, has been appointed associate professor of mathematics at the South Dakota State College.

Miss FRANCES B. HATCHER has been appointed associate professor of mathematics at Westhampton College, Richmond, Va.

S. L. BOOTHROYD, associate professor of mathematics and astronomy at the University of Washington, is spending the year at the Lick Observatory. He has been appointed to succeed Professor O. M. LELAND as professor of astronomy and geodesy at Cornell University and his duties are to commence in September, 1921.

Assistant Professor HILLEL HALPERIN, of the University of Arkansas, has been appointed to an associate professorship at the Texas Agricultural and Mechanical College.

Dr. TOBIAS DANTZIG, of Johns Hopkins University, has resigned to undertake mathematical research for an engineering company in New York City.

Popular Astronomy announces that at Carleton College, Dr. H. C. WILSON has been relieved for the present college year of his duties as professor of mathematics and astronomy and expects to spend part of the time in research at Mt. Wilson Observatory. Dr. E. A. FATH, formerly of Mt. Wilson Observatory, and later professor of astronomy in Beloit College, has been appointed in his place; he is also associated with the editorial work of *Popular Astronomy*.

At Purdue University, Assistant Professor W. A. ZEHRING has been promoted to an associate professorship, instructors C. T. HAZARD, C. K. ROBBINS, and Dr. G. H. GRAVES have been promoted to assistant professorships, and Dr. E. M. BERRY, W. H. FRAZIER, W. R. HARDMAN, and J. A. NEEDY have been appointed instructors.

At the University of North Carolina, Professor WILLIAM CAIN has been retired from active teaching after thirty-one years of service, and Professor ARCHIBALD HENDERSON (*1919*, 165, 208) succeeds him as head of the department of mathematics; Assistant Professors A. W. HOBBS and J. W. LASLEY, Jr., have been promoted to associate professorships.

At the University of Kentucky, Mr. W. E. PAYNE has been appointed instructor in mathematics.

Professor D. A. ROTHROCK has been elected dean of the College of liberal arts of Indiana University.

Mr. H. H. CONWELL, associate professor of mathematics in the University of Idaho, has resigned to accept a similar position in Beloit College.

Professor C. B. RIDGAWAY, head of the department of mathematics at the University of Wyoming, has retired after twenty-four years of service.

Professor J. M. TAYLOR, of Colgate University, [*1920*, 91] has retired after completing fifty-one years as teacher of mathematics in that institution.

In the department of mathematics of the University of Michigan, the following promotions are announced: Professors W. H. BUTTS and T. R. RUNNING to full professorships; Professors C. E. LOVE and T. H. HILDEBRANDT to associate professorships; Dr. L. J. ROUSE, Dr. A. L. NELSON, Dr. W. W. DENTON and Dr. R. B. ROBBINS to assistant professorships. Dr. ROBBINS has been granted leave of absence for the year, Mr. NORMAN ANNING, Mr. S. E. FIELD, Mr. K. W. HALBERT, Mr. G. D. JONES, Mr. J. N. LANDIS, and Mr. H. A. SIMMONS have been appointed instructors.

WILLIAM RINCK, professor of mathematics and registrar of Theological School and Calvin College, Grand Rapids, Michigan, and his eleven year old son were instantly killed by the overturning of an automobile on November 11, 1920. Professor Rinck was born October 6, 1877, at Rotterdam, Netherlands. While yet in his early teens America became his home. He was a graduate student in mathematics at the University of Michigan, 1900–1904, and at the University of Chicago, 1905–1906. He had been connected with Calvin College since that time.

Professor E. W. STANTON, fifty years a member of the teaching staff of Iowa State College, died September 12, 1920, at the age of seventy years.

Dr. K. F. W. ROHN, professor of mathematics at the University of Leipzig since 1905, died on August 4, 1920, aged sixty-five years. He is, perhaps, best known to mathematicians through the three volume work (in its third edition, 1906) entitled *Lehrbuch der darstellenden Geometrie* (1893–1896; fourth edition, 1913–1916), which he prepared, after the appearance of the first volume, in collaboration with Dr. E. Papperitz.

Dr. R. H. WEBER, son of the late Heinrich M. Weber (affectionately remembered by American mathematicians), and ordinary honorary professor of applied mathematics at the University of Rostock, died in August, 1920, aged forty-six years. He was appointed extraordinary professor at the University of Rostock in 1907. It will be recalled that he was the author of the sections in Weber and Wellstein, *Encyklopädie der Elementar-Mathematik*, dealing with "Analytische Statik," "Dynamik," and "Elektrische und magnetische Kraftlinien."

ARTHUR SEARLE, Phillips professor of astronomy, emeritus, at Harvard University since 1912, died on October 24, aged eighty-three years. An Englishman by birth but brought to this country at an early age, he and his brother George Mary, also an astronomer (1839–1918), graduated from Harvard within a year of one another. He was assistant at the Harvard Observatory 1869–83, assistant professor of astronomy 1883–87, and Phillips professor of astronomy

1887–1912. He wrote numerous astronomical papers, and a book entitled *Outlines of Astronomy*, 1874 (second edition, 1875); most of several volumes of the *Annals of the Harvard Observatory* (8, 1876; 62, parts 1, 2, 1907, 1911; 65–66, parts 1910; 67; 1912; 77, 1914) were prepared under his direction or written by him. In his little volume *Essays I–XXX*, published in 1910, the discussions are philosophical; essay IV deals with "Consciousness," essay VIII with "Mind and matter," essays XII–XVII with "Space and time," and essay XXX with "Mental diversities."

In addition to the previously listed [*1920*, 440] sixteen doctorates with mathematics as a major subject, conferred by American universities during the academic year 1919–1920, the following should be listed: H. R. BRAHANA, Princeton, "Curves on surfaces"; E. S. HAMMOND, Princeton, "Periodic conjugate nets of curves."

At the meeting of the American Mathematical Society, held at Columbia University on October 30, 1920, twelve papers were presented, the following speakers being also members of the Association: A. A. BENNETT, R. L. BORGER, O. E. GLENN, T. H. GRONWALL, D. JACKSON, E. KASNER, ELIZABETH LE STOURGEON, J. LIPKA, H. S. VANDIVER.

At the meeting of the National Academy of Science, at Princeton University on November 17, the following papers were read by members of the Association: "Equipartition of Energy" by E. B. WILSON; "Einstein gravitational fields: orbits and light rays" by E. KASNER; "Knots and Riemann spaces" by J. W. ALEXANDER; "The map coloring problem" by PHILIP FRANKLIN.

At the meeting of the mathematics section of the Virginia State Teachers' Association at Richmond, Va., on November 24, 1920, T. McN. SIMPSON, professor of mathematics at Randolph-Macon College, was elected president for the ensuing year. He delivered a paper on "Mathematics the common denominator of the sciences" which was published in the *Texas Mathematics Teachers' Bulletin*, November, 1920.

At the eighteenth annual meeting of The Association of Teachers of Mathematics in New England, held at the Massachusetts Institute of Technology, December 4, 1920, members of the Association gave the following addresses: "The mathematics of insurance" by Professor C. H. CURRIER; "Graphical methods of computation" by Professor JOSEPH LIPKA.

The Society of American Field Service Fellowships for French Universities offers fellowships for 1921–22, not to exceed twenty-five in number, and each of the value of $200 plus 10,000 francs. The awards are to be made early in 1921. Further details may be obtained on application to the secretary, Dr. I. L. Kandel, 522 Fifth Avenue, New York City.

A Royal medal has been awarded by the Royal Society to Professor G. H. Hardy, of Oxford University, for his researches in pure mathematics, particularly in the analytical theory of numbers and allied subjects.

Professor Adolf Hurwitz's mathematical library, containing 420 volumes of mathematical works, 370 volumes of periodicals, and 5000 reprints, is offered for sale. Those interested may see a detailed catalogue on applying to Mrs. Hurwitz, Bächtoldstrasse, Zürich, or to the editors of *Acta Mathematica*, Stockholm.

THE NOVEMBER MEETING OF THE MISSOURI SECTION.

The fourth regular meeting of the Missouri Section was held at the Kansas City Junior College, Kansas City, Missouri, on November 13. The single session was held in the morning, the chairman being Professor W. A. LUBY.

The attendance was twenty-four, including the following ten members of the Association:

A. C. Andrews, Minnie W. Caldwell, A. Davis, B. F. Finkel, R. R. Fleet, E. R. Hedrick, W. A. Luby, P. R. Rider, P. Robertson, Eula A. Weeks.

Professor E. R. Hedrick, chairman of the committee appointed[1] by the Missouri Section of the Association to consider the report of the National Committee, presented to the Section the report of the National Committee concerning the junior high schools. The opinions of the committee were, in general, favorable to the recommendations made by the National Committee but a considerable number of detailed remarks and suggestions were made which will be transmitted to the National Committee for their consideration. The only one of these which is far-reaching enough to deserve mention here is the recommendation to the National Committee that the work in demonstrative geometry ought not to be included in the junior high school. The committee was entirely in favor of leading up to demonstrative geometry but felt the work in the junior high schools should stop short of actual demonstrative work.

Mr. Davis stated to the meeting the objects of the National Council of Teachers of Mathematics organized at Cleveland, Ohio, February 24, 1920. The Section voted to become an institutional member of the Council.

The following new officers were elected: Chairman, Professor LOUIS INGOLD; Vice-chairman, Professor R. R. FLEET.

In accordance with previous arrangements the next regular meeting of the Section will be held at the University of Missouri, November 25–26, 1921, in connection with the meeting of the southwestern section of the American Mathematical Society.

The following four papers were read:

(1) "The relation of caustics to certain envelopes" by Professor O. DUNKEL;

(2) "A so-called Russian multiplication method" by Professor P. R. RIDER;

(3) "Sun-spot data and the methods of analysis applied" by Dr. D. ALTER, associate professor of astronomy, University of Kansas (invited);

(4) "The work of the National Committee on Mathematical Requirements" by Dr. EULA A. WEEKS.

[1] The other members of the committee are: Professor W. H. Zeigel, Professor R. R. Fleet, Miss Zoe Ferguson, Mr. Alfred Davis, Mr. Percival Robertson.